M-570, Mid-term Examination

Department of Mathematics,
University of Hael
Semester-I, 1431 H.

Time: 2 Hours **Max. Marks-30**

Note: *Attempt all questions*

Q.1. Answer the following

(i) Give an example of two subsets A, B of R^2 such that $A \cup B$ is connected but $A \cap B$ is not connected.

(ii) Consider the set

$$E = \{(x, y, z) \in R^3 : x^2 + y^2 = 1, \quad -1 \leq z \leq 1\}$$

and show that E is connected subset of R^3.

(iii) Is the topological space $R_f \times R_l$ connected? (justify your answer). Here R_f is R with finite complement topology and R_l is R with lower limit topology.

(iv) Let $x \in R_l$. Find the component C_x. Is C_x open subset of R_l? Is R_l locally connected?

Q.2. Answer the following

(i) If $f : [0, 1] \to [0, 1]$ is a continuous map, then show that there exists a point $x \in [0, 1]$ such that $f(x) = x$.

(ii) Let X be a topological space and $x \in X$ be a point in X. Show that the component C_x is a closed subset of X.

(iii) If X is locally connected space, then show that $C_x = Q_x$, where Q_x is the quasi-component of x.

(iv) Show that a homeomorphism $f : X \to Y$ gives a one-to-one correspondence between the set of components of X and the set of components of Y.

Q.3. (i) Let $U \subset R^n$ be an open subset and $f : U \to R^m$ be a function given in terms of components as

$$f = \left(f^1, .., f^m\right)$$

If each component function $f^i : U \to R$ is differentiable at $p \in U$, then show that f is differentiable at p.

(ii) Let $M(n, R)$ be the set of $n \times n$ real matrices and $f : M(n, R) \to M(n, R)$ be

$$f(X) = X^2,\ X \in M(n, R)$$

Show that f is differentiable at any $P \in M(n, R)$.

(iii) Show that the function $f : R^2 \to R$ defined by $f(x, y) = \sqrt{|xy|}$ is not differentiable at $0 = (0, 0) \in R^2$.