M-570, Final Examination

Department of Mathematics,

University of Hail

Semester-I, 1431 H.

Time: 3 Hours **Max. Marks-50**

Note: *Attempt all questions*

Q.1. Answer the following

(i) Consider the set

$$E = \{(x, y, z) \in R^3 : xy = 1, \quad 0 \leq z \leq 1\}.$$

Is the set E connected? For $p = (1, 1, 0) \in E$, find the component C_p. [2]

(ii) Let $X = D \cup \{(0, \frac{1}{2})\}$ be the subspace of R^2, where D is the deleted comb space. Find the path components of X. [3]

(iii) Give an example of a topological space X and a point $x \in X$ such that $C_x \neq Q_x$. [3]

(iv) Use the continuity of determinant function to check whether the general linear group $GL(n, R)$ is connected. [2]

Q.2. Answer the following

(i) Show that in a locally connected space X the components C_x are open subsets of X. Is the converse true? (Justify your answer by either giving a proof or a counter example). [3]

(ii) Show that the product $X \times Y$ is path connected if both X and Y are path connected. Is the converse true? [2]

(iii) If X is locally path connected space, then show that $P_x = C_x = Q_x$. [2]

(iv) Show that the subspaces $X = (0, 1) \cup [2, 3)$ and $Y = (0, 1) \cup (2, 3)$ of R are not homeomorphic. [3]

Q.3. (i) Let $U \subset R^n$ be an open subset and $p \in U$. If $f : U \to R^m$ is differentiable at p, then show that there is a **unique** $T \in L(R^n, R^m)$ such that $D_p f = T$. [3]

(ii) Let $U \subset R^n$ be an open subset and $p \in U$. If $f : U \to R^m$ is differentiable at p, find the matrix of $D_p f$. [3]

(iii) Assume that the function $f : R^2 \to R^3$ defined by $f(x, y, z) = \left(\frac{1}{2}(x^2 - y^2), xy, \frac{1}{2}(x^2 + y^2)\right)$ is differentiable at $p = (1, 1) \in R^2$ and find the rank of $D_p f$. [3]

Q.4. (i) Show that the function $f : R^2 \to R$ defined by

$$f(x, y) = e^{2x} \cos x$$

is differentiable at $p = (a, b) \in R^2$ and find $D_p f$. [4]

(ii) Let $U, V \subset R^n$ be an open subsets and $p \in U$. If $f : U \to V$ is one-one on-to differentiable at p and $f^{-1} : V \to U$ is differentiable at $q = f(p)$, then show that $D_p f$ is non-singular and find $(D_p f)^{-1}$. [3]

(iii) Assume that the function $f : R^2 \to R^2$ defined by $f(x, y) = \left(\frac{1}{2}(x^2 - y^2), \frac{1}{2}(x^2 + y^2)\right)$ is differentiable at $p = (1, 1) \in R^2$ and show that there exist neighbourhoods U and V of p and $q = (0, 1)$ respectively, such that $f : U \to V$ is one-one and on-to. Is $f : R^2 \to R^2$ one-one on-to? [3]

Q.5. (i) Let $U \subset S^2$ be defined by $U = \{(x, y, z) \in S^2 : x > 0\}$ and $B_1^2(0) \subset R^2$ be the open ball in R^2 of radius 1 centered at origin. Show that (U, φ) is a chart on S^2, where $\varphi : U \to B_1^2(0)$ is defined by

$$\varphi(x, y, z) = (y, z)$$

Also show that this chart belongs to the differentiable structure of S^2. [4]

(ii) Let $f : S^2 \to R$ be $f(x, y, z) = z$, and $p = (0, 0, 1).\in S^2$. Show that f is smooth at p. Choose a chart (U, φ) around p on S^2 with local coordinates x^1, x^2 and find the partial derivatives

$$\frac{\partial f}{\partial x^1}(p), \quad \frac{\partial f}{\partial x^2}(p)$$

[4]

(iii) Consider the functions $f, g : R^2 \to R$ defined by $f(x, y) = \frac{1}{2}(x^2 - y^2)$, $g(x, y) = \frac{1}{2}(x^2 + y^2)$. Show that there exists a chart (U, φ) around $p = (1, 1) \in R^2$ with local coordinates f, g. [3]