M-570 FINAL EXAMINATION

I-Semester, 1428-1429

Department of Mathematics (Women's Section)

College of Science, King Saud University

Time 3 Hours **Max. Marks-50**

Note: *Attempt all questions*

Q.1.(a) Show that the space $X = R \times R_l$ is not connected and find the component of the point $p = (0,0) \in X$.

(b) Show that R_f (R with cofinite topology) is path connected.

(c) Show that the set $GL(2,R)$ of 2×2 real matrices with nonzero determinant is not connected. Find the component of the identity matrix $I \in GL(2,R)$.

Q.2. (a) Give an example of a connected topological space that has three path components.

(b) Suppose X is locally connected. Show that each component of X is an open subset of X.

(c) Show that the subspaces $X = (0,1) \cup (2,3)$ and $Y = (0,1) \cup [2,3)$ of R are not homeomorphic.

Q.3. (a) Let $f : R^3 \to R^2$ be $f(x,y,z) = (xy, yz)$ and $p = (1,1,1)$. Assume that f is differentiable at p and find $(D_p f)(x,y,z)$. What is the dimension of the $Ker\,(D_p f)$?

(b) Show that the set $GL(n,R)$ of $n \times n$ real invertible matrices is an open subset of R^{n^2}. For the function $f : GL(n,R) \to R^{n^2}$, $f(X) = X + X^{-1}$, find $D_A f$ at $A \in GL(n,R)$.

(c) Let $f : R \to R^2$ and $g : R^2 \to R$ be

$$f(t) = (\cos t, \sin t), \quad g(x,y) = xy$$

and $p = (1,0)$. Find $(D_p f \circ g)(x,y)$, $(x,y) \in R^2$.

Q.4. (a) Show that the subspace $M = \left\{(x,y) \in R^2 : y^2 = x\right\}$ of R^2 a 1-dimensional topological manifold.

(b) Show that $U = \left\{(x,y,z) \in S^2 : z > 0\right\}$ and $\varphi : U \to R^2$, $\varphi(x,y,z) = (x,y)$ constitute a chart (U,φ) on S^2 and that this chart belongs to differentiable structure of S^2.

(c) Give an example of a smooth manifold M and a chart (U,φ) on M that does not belong to the differentiable structure of M.

Q.5.(a) Let $f : S^1 \to R$ be $f(x,y) = x - 2y$ and $p = (0,1) \in S^1$. Find $\frac{\partial f}{\partial t}(p)$, where t is local coordinate on a chart around p.

(b) For a smooth manifolds M and $p \in M$ show that for $X_p \in T_p M$, $X_p(c) = 0$ for a constant c.

(c) Let (U,φ) be a chart on a smooth manifold M with local coordinates $x^1, .., x^n$. Show that $\left\{dx_p^1, .., dx_p^n\right\}$ is a basis of the cotangent space $T_p^* M$.