M-574, Midterm Examination

II- SEMESTER-1430

Department of Mathematics

College of Science, King Saud University

Time 2 hours. **Max. Marks-30**

Note: *Attempt all questions*

Q.1. **(a)** Let $\{\varphi_t\}$ be the one parameter group of transformations on the smooth manifold M that generates $X \in \mathfrak{X}(M)$. For a smooth function $f : M \to R$ define $X^k(f) \in C^\infty(M)$ by

$$X^k(f) = \underset{k\text{-times}}{XX..X}(f)$$

Show that

$$\varphi_t^*(f)(p) = f(p) + t\,(Xf)\,(p) + \tfrac{t^2}{2!}\left(X^2 f\right)(p) + \tfrac{t^3}{3!}(X^3 f)(p) + ...$$

(b) For the one parameter group of transformations $\{\varphi_t\}$ on S^2 defined by

$$\varphi_t(x, y, z) = (x \cos t + y \sin t, y \sin t - x \sin t, z)$$

and a smooth function $f : S^2 \to R$ defined by $f(x, y, z) = xy + z$ and $p = (\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}) \in S^2$ compute first three terms of $\varphi_t^*(f)(p)$ in the expansion given in (a)

Q.2. **(a)** Let M be an n-dimensional smooth manifold and $X, Y \in \mathfrak{X}(M)$ induced by the one parameter groups of transformations $\{\varphi_t\}$ and $\{\psi_t\}$ respectively. Show that $[X, Y] = 0$ if and only if $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ holds for $t, s \in R$.

(b) Show that the set

$$O(n) = \left\{A \in GL(n, R) : AA^t = I\right\}$$

is a Lie-group and find the Lie-algebra $o(n)$ of the Lie-group $O(n)$.

Q.3. **(a)** Let $\mathcal{G}$ be the Lie-algebra of the Lie-group G. Show that the map $\phi : \mathcal{G} \to T_eG$ defined by $\phi(X) = X(e)$ is an isomorphism.

(b) Explain what do you mean by adjoint representation of a Lie-group. Show that for a connected Lie-group the center of the Lie-group G is the kernel of the adjoint representation.