M-574, FINAL EXAMINATION, II-SEMESTER (1430H)
DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE
KING SAUD UNIVERSITY

Time-3 hours Max. Marks-50

Note: *Attempt all questions*

Q.1. Let $X, Y, Z \in \mathfrak{X}(R^2)$ be

$$X = y^2\frac{\partial}{\partial x},\ Y = -xy\frac{\partial}{\partial y},\ Z = y^2\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}$$

(i) Show that X is a complete vector field
(ii) Show that Y is a complete vector field
(iii) Show that Z is not a complete vector field
(iv) What do you conclude from above?

Q.2 (i) Let $f : M \to N$ be a smooth map and $X, Y \in \mathfrak{X}(M)$ be f-related to $Z, W \in \mathfrak{X}(N)$ respectively. Show that $[X, Y]$ is f-related to $[Z, W]$.

(ii) Let $\{\varphi_t\}$ be a one-parameter group of transformations on M that induces $X \in \mathfrak{X}(M)$. Show that X is φ_t-related to itself.

Q.3. Let G be a Lie group with Lie algebra $\mathcal{G}$

(i) Show that the tangent space T_eG is isomorphic to $\mathcal{G}$

(ii) Show that the exponential map $\exp : \mathcal{G} \to G$ is a smooth map that is a diffeomorphism on a neighbourhood of $0 \in \mathcal{G}$

(iii) Let $X \in \mathcal{G}$ and $\sigma(t) = \exp tX$, $t \in R$. Show that $\varphi_t = R_{\sigma(t)}$ gives a one-parameter group of transformations $\{\varphi_t\}$ on G that induces X

Q.4.(i) Explain the action of a Lie group G on a smooth manifold M. Show that the orthogonal group $O(3)$ acts on S^2

(ii) If $\mu : G \times M \to M$ is the left action of the Lie group G on the smooth manifold M and $x_0 \in M$ is the fixed point of this action, then show that the map $\varphi : G \to Aut(T_{x_0}M)$ defined by $\varphi(g) = \left(d\mu_g\right)_{x_0}$ is the representation of G

Q.5.(i) Let $F \to E \xrightarrow{\pi} M$ be a smooth fiber bundle. Show that $\pi : E \to M$ is a submersion and that each fiber E_p, over $p \in M$ is a smooth manifold with tangent space $T_\xi E_p = Ker\,(d\pi_\xi)$, $\xi \in E_p$.

(ii) Let V be an n-dimensional vector space over R. Show that the smooth fiber bundle $V \to E \xrightarrow{\pi} M$ is a vector bundle if and only if its structure group is $Gl(n, R)$.