Differential Geometry and its Applications
Proc. Conf., in Honour of Leonhard Euler, Olomouc, August 2007

Some variational problems in geometry of submanifolds

Haizhong Li

Department of Mathematical Sciences, Tsinghua University,
Beijing, 100084, People's Republic of China
E-mail: hli@math.tsinghua.edu.cn

In this paper, we present a survey of some variational problems in geometry of submanifolds, which includes our recent research results in geometry of r-minimal submanifolds, geometry of *Willmore* submanifolds and variations of some parametric elliptic functional. We also propose some open problems at the end of paper.

Keywords: r-minimal submanifold, Willmore submanifold, r-anisotropic mean curvature, Wulff shape.

MS classification: 53C42, 53A10, 49Q10.

1. Introduction

This paper is to present our recent research results about variational problems in geometry of submanifolds and divides into six sections. Now we describe in more detail some of the contents of this paper, where we refer to the papers for the explanation of notations and undefined terms.

The section 1 is the introduction. The section 2 is preliminaries and is devoted to some formulas and notations of submanifolds in geometry of submanifolds. In section 3 we present the first and second variational formula of the volume of n-dimensional submanifolds in an $(n+p)$-dimensional manifold N^{n+p}. Besides, we give two important results of J. Simons. In section 4, we will recall our recent works in Ref. 4. We introduce the functional

$$J_r = \int_M F_r(S_0, S_2, \cdots, S_r) dv_g$$

where function F_r is a suitable function on M, and introduce the concepts of r-minimal submanifolds and stability. We study the stability of compact r-minimal submanifold in the unit sphere S^{n+p}. In section 5 we give some facts (see Refs. 10,16,17) about variational problems of Willmore functional on submanifolds. In particular, we recall the Euler-Lagrange equation of $W(x)$ for an n-dimensional submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}, give some typical examples of Willmore submanifolds and give integral inequality of Simons' type for n-dimensional closed Willmore submanifolds in S^{n+p}. In section 6 we consider a variation problem concerning certain parametric elliptic functional and collect some results of B. Palmer and He-Li. We state integral formula of Minkowski's type[8] and new characterizations of the Wulff shape.[9,22]

2. Preliminaries

Let(N^{n+p}, h) be an $n+p$-dimensional oriented smooth Riemannian manifold, and $x : M^n \to N^{n+p}$ be an n-dimensional submanifold of (N^{n+p}, h). We will agree on the following index convention:

$$1 \le i, j, k, \cdots \le n; \quad n+1 \le \alpha, \beta, \gamma, \cdots \le n+p; \quad 1 \le A, B, C, \cdots \le n+p$$

Let$\{e_A\}$ be a local orthonormal basis for TN^{n+p} with dual basis $\{\theta_A\}$ such that when restricted to M^n, $\{e_i\}$ is a local orthonormal basis for TM and $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of $x : M^n \to N^{n+p}$. Let $\{\omega_{AB}\}$ be the connection forms of (N^{n+p}, h), they are characterized by the following structure equations

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0 \tag{1}$$

$$\omega_{AB} = \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \bar{R}_{ABCD} \omega_C \wedge \omega_D \tag{2}$$

where$\bar{R}_{ABCD}$ are the components of the Riemannian curvature tensor of (N^{n+p}, h).

Now we restrict to a neighborhood of $x : M^n \to N^{n+p}$. Let θ_A, θ_{AB} be the restrictions of ω_A, ω_{AB} to M^n, then we have

$$\theta_\alpha = 0 \tag{3}$$

Taking its exterior derivative and by (1) we get

$$\sum_i \theta_{\alpha i} \wedge \theta_i = 0 \tag{4}$$

By Cartan's lemma we have

$$\theta_{i\alpha} = \sum_j h^\alpha_{ij} \theta_j, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{5}$$

We can define the second fundamental form B_{ij} and the mean curvature vector $\mathbb{H}$ of x as follows

$$B_{ij} := \sum_{\alpha} h^{\alpha}_{ij} e_{\alpha}, \qquad \mathbb{H} = \frac{1}{n}\sum_{i,\alpha} h^{\alpha}_{ii} := \frac{1}{n}\sum_{\alpha} H^{\alpha} e_{\alpha} \tag{6}$$

Let $S = |B|^2 = \sum_{i,j,\alpha} (h^{\alpha}_{ij})^2$ and $H = |\mathbb{H}|$ be the norm square of B and the mean curvature of $x : M^n \to N^{n+p}$, respectively. If we denote by R_{ijkl}, R_{ij}, R the Riemannian curvature tensor, Ricci curvature and scalar curvature of M, respectively. We have Gauss equations, Ricci equations and Codazzi equations as followings (see Refs. 3,10)

$$R_{ijkl} = \bar{R}_{ijkl} + \sum_{\alpha} (h^{\alpha}_{ik} h^{\alpha}_{jl} - h^{\alpha}_{il} h^{\alpha}_{jk}) \tag{7}$$

$$R_{\alpha\beta ij} = \bar{R}_{\alpha\beta ij} + \sum_{\alpha} (h^{\alpha}_{ik} h^{\beta}_{kj} - h^{\alpha}_{jk} h^{\beta}_{ki}), \tag{8}$$

$$h^{\alpha}_{ijk} - h^{\alpha}_{ikj} = \bar{R}_{\alpha ikj}, \tag{9}$$

where h^{α}_{ijk} is the covariant derivative of h^{α}_{ij}.

In particular, when the ambient space N^{n+p} is a space form $R^{n+p}(c)$, the Gauss equation, Ricci equation and Codazzi equation are (also see Refs. 14, 15)

$$R_{ijkl} = c(\delta_{ik}\delta_{ji} - \delta_{il}\delta_{jk}) + \sum_{\alpha} (h^{\alpha}_{ik} h^{\alpha}_{jl} - h^{\alpha}_{il} h^{\alpha}_{jk}), \tag{10}$$

$$R_{\alpha\beta ij} = \sum_{\alpha} (h^{\alpha}_{ik} h^{\beta}_{kj} - h^{\alpha}_{jk} h^{\beta}_{ki}), \tag{11}$$

$$h^{\alpha}_{ijk} = h^{\alpha}_{ikj}. \tag{12}$$

3. Minimal submanifolds in a Riemannian manifold

Let $x_0 : M \to N^{n+p}$ be an n-dimensional submanifolds in an $(n+p)$-dimensional manifold N^{n+p}, we assume, without loss of generality, that M is compact with (possibly empty) boundary. If otherwise, we will consider the variation with compact support.

Let $x : M \times R \to N^{n+p}$ be a smooth variation of x_0 such that $x(\cdot, t) = x_0$ and $dx_t(TM) = dx_0(TM)$ on ∂M for each small t, where $x_t(p) = x(p, t)$. Let $V(t)$ be the volume functional of $x_t(M)$, i.e.

$$V(t) = \int_M \theta_1 \wedge \cdots \wedge \theta_n. \tag{13}$$

We have (see Refs. 3,21)

$$V'(t) = -n \int_M \sum_\alpha H^\alpha a_\alpha \theta_1 \wedge \cdots \wedge \theta_n, \tag{14}$$

where $\sum\limits_\alpha a_\alpha e_\alpha$ is the normal variation vector field.

Definition 3.1. Let $x_0 : M \to N^{n+p}$ be an n-dimensional submanifold in an $(n+p)$-dimensional manifold N^{n+p}, if

$$\vec{H} = \frac{1}{n} \sum_\alpha H^\alpha e_\alpha \equiv 0, \tag{15}$$

we call M be a minimal submanifold.

Proposition 3.1 (Refs. 3,27). *Let $x_0 : M \to N^{n+p}$ be an n-dimensional submanifold in an $(n+p)$-dimensional manifold N^{n+p}, then we have*

$$V''(0) = -\int_M \sum_\alpha a_\alpha (\Delta^\perp a_\alpha + \sum_\beta \sigma_{\alpha\beta} a_\beta + \sum_{i=1}^{n} \sum_{\beta=n+1}^{n+p} \tilde{R}_{\alpha i \beta j} a_\beta) \theta_1 \wedge \cdots \wedge \theta_n, \tag{16}$$

where $\tilde{R}_{\alpha i \beta j}$ is the components of Riemannian curvature tensor, $\sigma_{\alpha\beta} = \sum\limits_{i,j} h^\alpha_{ij} h^\beta_{ij}$, and $\Delta^\perp a_\alpha$ is the Laplacian of a_α in the normal bundle.

Definition 3.2. Let $x : M \to N^{n+p}$ be an n-dimensional submanifold in N^{n+p}. If for any normal variational vector $W = \sum\limits_\alpha a_\alpha e_\alpha$, we have $V''(0) \geq 0$, we call M is stable.

Theorem 3.1 (J. Simons, Ref. 27). *There exists no any n-dimensional closed stable minimal submanifolds in an $(n+1)$-dimensional unit sphere S^{n+p}.*

Example 3.1 (see Refs. 1,13). Clifford tori:

$$C_{m,n-m} = S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}}) \quad , \quad 1 \leq m \leq n-1$$

are minimal hypersurfaces in S^{n+1}.

Theorem 3.2 (Refs. 1,13,27). *Let M be an n-dimensional ($n \geq 2$) closed minimal submanifold in $(n+p)$-dimensional unit sphere S^{n+p}. Then we have*

$$\int_M S \left(\frac{n}{2 - 1/p} - S \right) dv \leq 0. \tag{17}$$

In particular, if

$$0 \le S \le \frac{n}{2 - 1/p}, \tag{18}$$

then either $S \equiv 0$ *and* M *is totally geodesic, or* $S \equiv \dfrac{n}{2-1/p}$*. In the latter case, either* $p = 1$ *and* M *is a Clifford torus* $C_{m,n-m}$*; or* $n = 2$, $p = 2$ *and* M *is the Veronese surface.*

4. r-minimal submanifolds in a space form

We recall the definition of the generalized Kronecker symbols. If $i_1, \cdots, i_r$ and $j_1, \cdots, j_r$ are integers between 1 and n, then $\delta^{i_1 \cdots i_r}_{j_1 \cdots j_r}$ is $+1$ or -1 according as the i's are distinct and the j's are an even or odd permutation of the $i's$, and is 0 in all other cases.

Let (M, g) be an n-dimensional submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$, and B be the second fundamental form of M. Suppose $\{e_i\}$ is a local orthonormal basis for TM with dual basis $\{\theta_i\}$ and $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of $x : M^n \to R^{n+p}(c)$, (B_{ij}) is the matrix with respect to the frame $\{e_1, \cdots, e_n\}$ on M. Then for any even integer $r \in \{0, 1, \cdots, n-1\}$, we introduce r-th mean curvature function S_r and $(r+1)$-th mean curvature vector field $\vec{S}_{r+1}$ as follows (see Refs. 4,5,25,26:

$$\begin{aligned}
S_r &= \frac{1}{r!} \sum_{\substack{i_1 \cdots i_r \\ j_1 \cdots j_r}} \delta^{i_1 \cdots i_r}_{j_1 \cdots j_r} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}}, B_{i_r j_r} \rangle \\
&= \frac{1}{r} \sum_{i,j,\alpha} T_{r-1}{}^{\alpha}_{\ ij} h^{\alpha}_{ij}, \\
\vec{S}_{r+1} &= \frac{1}{(r+1)!} \sum_{\substack{i_1 \cdots i_{r+1} \\ j_1 \cdots j_{r+1}}} \delta^{i_1 \cdots i_{r+1}}_{j_1 \cdots j_{r+1}} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}}, B_{i_r j_r} \rangle B_{i_{r+1} j_{r+1}} \\
&= \frac{1}{r+1} \sum_{i,j,\alpha} T_{r}{}^{i}_{j} h^{\alpha}_{ij} e_\alpha, \\
S_r &= \binom{n}{r} H_r, \quad \vec{S}_{r+1} = \binom{n}{r+1} \vec{H}_{r+1},
\end{aligned}$$

where $\binom{n}{r}$ being the binomial coefficient.

Besides, we define the following $(0,2)$-tensor T_r for $r \in \{1, \cdots, n-1\}$:

- If r is even, we set

$$T_r = \frac{1}{r!} \sum_{\substack{i_1 \cdots i_r i \\ j_1 \cdots j_r j}} \delta^{i_1 \cdots i_r i}_{j_1 \cdots j_r j} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}}, B_{i_r j_r} \rangle \theta_i \otimes \theta_j$$
$$= \sum_{i,j} T_{r\,j}^{\,i} \theta_i \otimes \theta_j.$$

- If $r-1$ is odd, we set

$$T_{r-1} = \frac{1}{(r-1)!} \sum_{\substack{i_1 \cdots i_{r-1} i \\ j_1 \cdots j_{r-1} j}} \delta^{i_1 \cdots i_{r-1} i}_{j_1 \cdots j_{r-1} j} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-3} j_{r-3}}, B_{i_{r-2} j_{r-2}} \rangle$$
$$\cdot B_{i_{r-1} j_{r-1}} \theta_i \otimes \theta_j$$
$$= \sum_{i,j,\alpha} T_{r-1\ ij}^{\ \alpha} \theta_i \otimes \theta_j e_\alpha,$$

By convention, we put $H_0 = S_0 = 1$, T_0 =identity.

Let $x : M \to R^{n+p}(c)$ be an n-dimensional compact submanifold in $R^{n+p}(c)$. Assume that r is even and $r \in \{0, 1, \cdots, n-1\}$, , in Ref. 4 the authors introduce a curvature integral J_r for r even and $r \in \{0, 1, \cdots, n-1\}$

$$J_r = \int_M F_r(S_0, S_2, \cdots, S_r) dv$$

where the function F_r are defined inductively by

$$\begin{cases} F_0 = 1 \\ F_r = S_r + \frac{(n-r+1)c}{r-1} F_{r-2}, \quad \text{for} \ \ 2 \le r \le n-1. \end{cases}$$

Theorem 4.1 (Ref. 4, the first variational formula). *Let M be an n-dimensional compact, possibly with boundary, submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. Assume that r is even and $r \in \{0, 1, \cdots n-1\}$, then we have*

$$J_r{}'(t) = -(r+1) \int_{M_t} \langle \vec{S}_{r+1}, V \rangle dv_{g_t}$$

where V is the variational vector field.

Definition 4.1 (Ref. 4). Let M be an n-dimensional submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. Assume that r is even and $r \in \{0, 1, \cdots, n-1\}$, we call x to be r-minimal if its $(r+1)$-th mean curvature vector $\vec{S}_{r+1}$ vanishes on M.

Theorem 4.2 (Ref. 4, the second variational formula). *Let M be an n-dimensional compact r-minimal submanifold in $R^{n+p}(c)$ for some even integer $r \in \{0, 1, \cdots, n-1\}$, we have*

$$J''(0) = -\int_M \sum_\alpha V_\alpha \{ \frac{1}{r-1} \sum_{\substack{i_{r-1} i_r i \\ j_{r-1} j_r j \\ \beta}} T_{r-2}{}^{i_{r-1} i_r i}_{j_{r-1} j_r j} h^{\ \alpha}_{i_{r-1} j_{r-1}} h^{\ \beta}_{i_r j_r} V_{\beta,ij}$$
$$+ \sum_{i,j} T_r{}^i_j V_{\alpha,ij} + c \cdot (n-r)(S_r V_\alpha + \sum_{i,j,\beta} T_{r-1}{}^{\alpha}_{ij} h^\beta_{ij} V_\beta)$$
$$- (r+1) \sum_{i,j,\beta} T_{r+1}{}^{\alpha}_{ij} h^\beta_{ij} V_\beta \} dv$$

Definition 4.2 (Ref. 4). Assume that r is even and $r \in \{0, 1, \cdots, n-1\}$, and let M be an n-dimensional compact r-minimal submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. If $J_r''(0) \geq 0$ for arbitrary variations, we call M to be stable.

Remark 4.1. From definition 4.1, we know that concept of 0-minimal submanifolds is the concept of minimal submanifolds.

Theorem 4.3 (Ref. 4). *Assume that r is even and $r \in \{0, 1, \cdots, n-1\}$. If S_r is positive, then there exists no any closed stable r-minimal submanifold in the unit sphere S^{n+p}.*

Remark 4.2. Noting when $r = 0$, i.e. $S_0 = 1$, it is obvious to see that our Theorem 4.3 reduces to J. Simons' Theorem 3.2.

Theorem 4.4 (Ref. 4). *Assume that r is even and $r \in \{1, \cdots, n-1\}$. If $S_r \geq 0$, then any closed stable r-minimal hypersurface in unit sphere S^{n+1} must be a geodesic sphere.*

5. Willmore submanifolds in a Riemannian manifold

Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold and $x : M \to N^{n+p}$ an isometric immersion of an n-dimensional Riemannian manifold M. Willmore functional is the following non-negative functional:(see Refs. 2,24 or Ref. 28)

$$W(x) := \int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv.$$

where $S = \sum_{\alpha,i,j} (h^\alpha_{ij})^2$ and H are respectively the norm square of the second fundamental form and the mean curvature of the immersion x, dv is the volume element of M. An immersion $x : M \to N^{n+p}$ is called Willmore if it is an extremal submanifold of the Willmore functional. The famous Willmore conjecture can be stated in a equivalent way as that

$$W(x) \geq 4\pi^2$$

holds for all immersed tori $x : M \to S^3$ (see Ref. 30).

It is well known (see Refs. 2,24 or Ref. 28) that Willmore functional is invariant under conformal transformations of N^{n+p}. In Refs. 16,24,28, the authors calculated the first variation formula of Euler-Lagrangian equation of $W(x)$ for an n-dimensional submanifold in $R^{n+p}(c)$. In Ref. 6, the authors calculated the second variation formula of $W(x)$ for Willmore submanifolds $x : M^n \to \mathbb{S}^{n+p}$ without umbilic points in terms of Möbius geometry and gave many examples of Willmore submanifolds.

In Ref. 10 the authors calculated the Euler-Lagrangian equation for the critical points of $W(x)$ for the most general case.

Theorem 5.1 (Ref. 10). *The variation of the Willmore functional depends only on the normal component of the variation vector field. A submanifold $x : M \to N^{n+p}$ is a Willmore submanifold if and only if*

$$\begin{aligned}
&\rho^{n-2}\Big[\sum_{i,j,k,\beta} h^\beta_{ij}h^\beta_{ik}h^\alpha_{kj} + \sum_{i,j,\beta}\tilde{R}_{\beta i\alpha j}h^\beta_{ij} - \sum_{i,j,\beta} H^\beta h^\beta_{ij}h^\alpha_{ij} \\
&- \sum_{i,\beta} H^\beta \tilde{R}_{\beta i\alpha i} - \rho^2 H^\alpha\Big] + \sum_{i,j}\Big\{2\big(\rho^{n-2}\big)_{,i}h^\alpha_{ijj} + \big(\rho^{n-2}\big)_{,i,j}h^\alpha_{ij} \\
&+ \rho^{n-2}h^\alpha_{ijij}\Big\} - H^\alpha\Delta(\rho^{n-2}) - \rho^{n-2}\Delta^\perp H^\alpha - 2\sum_i \big(\rho^{n-2}\big)_{,i}H^\alpha_{,i} = 0, \\
&\quad n+1 \leq \alpha \leq n+p.
\end{aligned} \tag{19}$$

where $\tilde{R}_{ABCD}$ are the components of the Riemannian curvature tensor of (N^{n+p}, h).

In particular, when $N^{n+p} = R^{n+p}(c)$, we have

Theorem 5.2 (Refs. 16,24). *A submanifold $x : M^n \to R^{n+p}(c)$ is a Willmore submanifold if and only if*

$$\begin{aligned}
&\rho^{n-2}\Big[\sum_{i,j,k,\beta} h^\beta_{ij}h^\beta_{ik}h^\alpha_{kj} - \sum_{i,j,\beta} H^\beta h^\beta_{ij}h^\alpha_{ij} - \rho^2 H^\alpha\Big] \\
&+ (n-1)H^\alpha\Delta\big(\rho^{n-2}\big) + 2(n-1)\sum_i \big(\rho^{n-2}\big)_{,i}H^\alpha_{,i} + (n-1)\rho^{n-2}\Delta^\perp H^\alpha \\
&- \sum_{i,j}\big(\rho^{n-2}\big)_{,i,j}\big(nH^\alpha\delta_{ij} - h^\alpha_{ij}\big) = 0, \quad n+1 \leq \alpha \leq n+p.
\end{aligned} \tag{20}$$

Example 5.1 (Ref. 29). Every minimal surface in $R^{n+p}(c)$ is Willmore. We note that there are much more abundance of non-minimal Willmore surfaces in $R^{2+p}(c)$, see, e.g. Refs. 20,23, among many others.

Example 5.2 (Refs. 6,24). Every n-dimensional (≥ 3) minimal and Einstein submanifold in $N^{n+p}(c)$ is Willmore.

Example 5.3 (Refs. 6,16). $W_{n_1,\cdots,n_{p+1}} = S^{n_1}(a_1) \times \cdots \times S^{n_{p+1}}(a_{p+1})$ is an n-dimensional Willmore submanifold in $S^{n+p}(1)$, where $n = n_1 + \cdots + n_{p+1}$ and a_i are defined by

$$a_i = \sqrt{\frac{n - n_i}{np}}, \qquad i = 1, \cdots, p+1.$$

Furthermore, $W_{n_1,\cdots,n_{p+1}}$ is a minimal submanifold in $S^{n+p}(1)$ if and only if it is Einstein with

$$n_1 = \cdots = n_{p+1} = \frac{n}{p+1}, a_i = \sqrt{\frac{1}{p+1}}.$$

Example 5.4 (Refs. 6,16,17). Willmore tori:

$$W_{m,n-m} = S^m(\sqrt{\frac{n-m}{n}}) \times S^{n-m}(\sqrt{\frac{m}{n}}), \quad 1 \leq m \leq n-1$$

are Willmore hypersurfaces in $S^{n+1}(1)$.

Example 5.5 (Ref. 11). Define the Lagrangian sphere $\Psi : \mathbb{S}^n \to \mathbb{C}^n$ by

$$\begin{aligned}&\Psi(x_1, \cdots, x_n, x_{n+1})\\ &= \frac{2\sqrt{\frac{n-1}{2n}}\, e^{i\beta(x_{n+1})}}{[(1+x_{n+1})^{\sqrt{\frac{2n}{n-1}}} + (1-x_{n+1})^{\sqrt{\frac{2n}{n-1}}}]^{\sqrt{\frac{n-1}{2n}}}} \cdot (x_1, \cdots, x_n),\end{aligned}$$

$$\beta(x_{n+1}) = \sqrt{\frac{2(n-1)}{n}} \arctan(\frac{(1+x_{n+1})^{\sqrt{\frac{n}{2(n-1)}}} - (1-x_{n+1})^{\sqrt{\frac{n}{2(n-1)}}}}{(1+x_{n+1})^{\sqrt{\frac{n}{2(n-1)}}} + (1-x_{n+1})^{\sqrt{\frac{n}{2(n-1)}}}}).$$

Then Ψ is a Lagrangian Willmore submanifold. We call Ψ as *the Lagrangian Willmore sphere*. We note that when $n = 2$, Ψ is *Whitney sphere*

Theorem 5.3 (Refs. 16,17). *Let M be an n-dimensional ($n \geq 2$) closed Willmore submanifold in $(n+p)$-dimensional unit sphere S^{n+p}. Then we have*

$$\int_M \rho^n \left(\frac{n}{2 - 1/p} - \rho^2 \right) dv \leq 0. \tag{21}$$

In particular, if

$$0 \le \rho^2 \le \frac{n}{2 - 1/p}, \tag{22}$$

then either $\rho^2 \equiv 0$ *and* M *is totally umbilic, or* $\rho^2 \equiv \frac{n}{2-1/p}$*. In the latter case, either* $p = 1$ *and* M *is a Willmore torus* $W_{m,n-m}$*; or* $n = 2$, $p = 2$ *and* M *is the Veronese surface.*

For $n = 2$, the following result was proved by Li[18] (also see Li-Simon[19])

Theorem 5.4 (Refs. 18,19). *Let* M *be a closed Willmore surface in an* $(2 + p)$*-dimensional unit sphere* S^{2+p}*. Then we have*

$$\int_M \rho^2(2 - \frac{3}{2}\rho^2)dv \le 0. \tag{23}$$

In particular, if

$$0 \le \rho^2 \le \frac{4}{3}, \tag{24}$$

then either $\rho^2 = 0$ *and* M *is totally umbilic, or* $\rho^2 = \frac{4}{3}$*. In the latter case,* $p = 2$ *and* M *is the Veronese surface.*

Remark 5.1. In Ref. 21, the authors give some examples of 3-dimensional Lagrangian Willmore submanifolds in S^6. In Ref. 7, the authors study the variational problem of the functional $F(x) = \int_M (S - nH^2)dv$ for submanifold $x : M^n \to S^{n+p}$ and get an integral inequality of Simons' type.

6. Variations of Some Parametric Elliptic Functional

Let $F\colon S^n \to \mathbb{R}^+$ be a smooth function which satisfies the following convexity condition:

$$(D^2F + F1)_x > 0, \quad \forall x \in S^n, \tag{25}$$

where D^2F denotes the intrinsic Hessian of F on S^n and 1 denotes the identity on T_xS^n, > 0 means that the matrix is positive definite. We consider the map

$$\begin{aligned} \phi\colon S^n &\to \mathbb{R}^{n+1} \\ x &\to F(x)x + (\mathrm{grad}_{S^n} F)_x, \end{aligned}$$

its image $W_F = \phi(S^n)$ is a smooth, convex hypersurface in $\mathbb{R}^{n+1}$ called the Wulff shape of F (see Refs. 8,9,12,22).

Now let $X\colon M \to \mathbb{R}^{n+1}$ be a smooth immersion of a compact, orientable hypersurface without boundary. Let $\nu\colon M \to S^n$ denotes its Gauss map, that is, ν is an unit inner normal vector of M.

Let $A_F = D^2F + F1$, $S_F = -A_F \circ \mathrm{d}\nu$. S_F is called the F-Weingarten operator, and the eigenvalues of S_F are called anisotropic principal curvatures. Let σ_r be the elementary symmetric functions of the anisotropic principal curvatures $\lambda_1, \lambda_2, \cdots, \lambda_n$:

$$\sigma_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} \quad (1 \le r \le n).$$

We set $\sigma_0 = 1$. The r-anisotropic mean curvature M_r is defined by $M_r = \sigma_r / C_n^r$.

In Ref. 8, we obtained the following integral formulas of Minkowski's type for closed hypersurfaces in $\mathbb{R}^{n+1}$.

Theorem 6.1 (Ref. 8). *Let $X\colon M \to \mathbb{R}^{n+1}$ be an n-dimensional closed hypersurface, $F\colon S^n \to \mathbb{R}^+$ be a smooth function which satisfies (25), then we have the following integral formulas of Minkowski type hold:*

$$\int_M (FM_r + M_{r+1}\langle X, \nu\rangle)\mathrm{d}A_X = 0, \quad r = 0, 1, \cdots, n-1. \tag{26}$$

By use of above integral formulas of Minkowski type, we prove the following new characterizations of the Wulff shape:

Theorem 6.2 (Ref. 8). *Let $X\colon M \to \mathbb{R}^{n+1}$ be an n-dimensional closed hypersurface, $F\colon S^n \to \mathbb{R}^+$ be a smooth function which satisfies (6.1), and $M_1 = const$ and $\langle X, \nu\rangle$ has fixed sign, then up to translations and homotheties, $X(M)$ is the Wulff shape.*

Theorem 6.3 (Ref. 8). *Let $X\colon M \to \mathbb{R}^{n+1}$ be an n-dimensional closed hypersurface, $F\colon S^n \to \mathbb{R}^+$ be a smooth function which satisfies (25). If $M_1 = const$ and $M_r = const$ for some r, $2 \le r \le n$, then up to translations and homotheties, $X(M)$ is the Wulff shape.*

Theorem 6.4 (Ref. 8). *Let $X\colon M \to \mathbb{R}^{n+1}$ be an n-dimensional closed convex hypersurface, $F\colon S^n \to \mathbb{R}^+$ be a smooth function which satisfies (6.1). If $\frac{M_r}{M_k} = const$ for some k and r, with $0 \le k < r \le n$, then then up to translations and homotheties, $X(M)$ is the Wulff shape.*

For each r, $0 \le r \le n$, we set

$$\mathscr{A}_r = \int_M F(\nu)\sigma_r dA_X. \tag{27}$$

Suppose σ_r is positive on M, and consider those hypersurfaces which are critical points of the functional $\mathscr{A}_r$ restricted to those hypersurfaces with the same $\mathscr{A}_{r-1}$, where $r \geq 1$. By a standard argument involving Lagrange multipliers, this means we are considering critical points of the functional

$$\mathscr{F}_{r;\Lambda} = \mathscr{A}_r + \Lambda \mathscr{A}_{r-1}, \tag{28}$$

where Λ is a constant. In Ref. 9 the authors show that the Euler-Lagrange equation of $\mathscr{F}_{r;\Lambda}$ is:

$$(r+1)\sigma_{r+1} + \Lambda r \sigma_r = 0. \tag{29}$$

So the critical points are just hypersurfaces with $M_{r+1}/M_r = \text{const}$.

We call a critical immersion X stable if and only if the second variation of $\mathscr{A}_r$ (or equivalently of $\mathscr{F}_{r;\Lambda}$) is non-negative for all variations of X preserving $\mathscr{A}_{r-1}$. We have the following theorems.

Theorem 6.5 (Ref. 9). *For each r, $1 \leq r \leq n-1$, the Wulff shape W_F is a stable critical immersion.*

Theorem 6.6 (Ref. 9). *Suppose $1 \leq r \leq n-1$. Let $X \colon M \to \mathbb{R}^{n+1}$ be a smooth immersion of an oriented closed, stable critical point of $\mathscr{A}_r$ for all variations of X preserving $\mathscr{A}_{r-1}$. Then, up to translations and homotheties, $X(M)$ is the Wulff shape.*

In Ref. 22, B. Palmer considered a variational problem of the functional $\mathscr{A}_0$ restricted to those hypersurfaces preserving the enclosed volume. He studied the first and second variations of the functional $\mathscr{A}_0$ preserving volume and showed that, up to homothety and translation, the only closed, oriented, stable critical point is the Wulff shape.

At the end of this paper, we would like to propose the following open problems:

Problem 1. *Assume that r is even and $r \in \{2, \cdots, n-1\}$. Is any closed stable r-minimal hypersurface in unit sphere S^{n+1} a geodesic sphere?*

Problem 2. *Let $x : M \to S^{n+1}$ be a n-dimensional Willmore hypersurface ρ^2 =constant, is the value of ρ^2 the discrete?*

Problem 3. *Is any topological sphere in R^3 with constant anisotropic mean curvature the Wulff shape?*

Problem 4. *Is any n-dimensional ($n \geq 2$) closed embedded hypersurface in R^{n+1} with constant anisotropic mean curvature the Wulff shape?*

Acknowledgments

The author would like to thank Professor Oldrich Kowalski, chairman of Section A: Riemannian Geometry and Submanifolds, in "DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS-the 10th International Conference Dedicated to the 300th Birthday of Leonhard Euler August 27 - 31, 2007, Olomouc, Czech Republic", for inviting him to give a plenary talk in Section A. This work is partially supported by grant No. 10531090 of NSFC and by SRFDP.

References

1. S.S. Chern, M. Do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, *In: Functional Analysis and Related Fields* ((F.Brower, Ed.) Springer-Verlag, Berlin, 1970) 59–75.
2. B.Y. Chen, Some conformal invariants of submanifolds and their applications, *Boll. Un. Mat. Ital.* **10** (1974) 380–385.
3. S.S. Chern, *Minimal Submanifolds in a Riemannian Manifold (mimeographed)* (University of Kansas, Lawrence, 1968).
4. L.F. Cao and H. Li, r-minimal submanifolds in space forms, *Ann. Global Anal. Geom.* **32** (2007) 311–341.
5. J.F. Grosjean, Upper bounds for the first eigenvalue of the Laplacian on compact submanifolds, *Pacific J. Math.* **206** (2002) 93–112.
6. Z. Guo, H. Li and C.P. Wang, The second variational formula for Willmore submanifolds in S^n, *Results in Math.* **40** (2001) 205–225.
7. Z. Guo and H. Li, A variational problem for submanifolds in a sphere, *Monatshefte Mathematik* **152** (2007) 295–302.
8. Y.J. He and H. Li, Integral formula of Minkowski type and new characterization of the Wulff shape, 2007, to appear in *Acta Math. Sinica*; arXiv:math.DG/0703187.
9. Y.J. He and H. Li, A new variational characterization of the Wulff shape, to appear in *Diff. Geom. Appl.*, 2007.
10. Z.J. Hu and H. Li, Willmore submanifolds in Riemannian Manifolds, 251-275, *In: Proceedings of the Workshop Contem. Geom. and Related Topics* (Belgrand, Yugoslavia, May, 2002) 15-21.
11. Z.J. Hu and H. Li Willmore Lagrangian spheres in the complex Euclidean space C^n, *Ann. Global Anal. Geom.* **25** (2004) 73–98.
12. M. Koiso and B. Palmer, Geometry and stability of surfaces with constant anisotropic mean curvature, *Indiana Univ. Math. J.* **54** (2005) 1817–1852.
13. H.B. Lawson, Local rigidity theorems for minimal hypersurfaces, *Ann. of Math.* **89** (1969) 187–197.
14. H. Li, Hypersurfaces with constant scalar curvature in space forms, *Math. Ann.* **305** (1996) 665–672.
15. H. Li, Global rigidity theorems of hypersurface, *Ark. Math.* **35** (1997) 327–351.

16. H. Li, Willmore submanifolds in a sphere, *Math. Research Letters* **9** (2002) 771–790.
17. H. Li, Willmore hypersurfaces in a sphere, *Asian J. Math.* **5** (2001) 365–378.
18. H. Li , Willmore surfaces in S^n, *Ann. Global Anal. Geom.* **21** (2002) 203–213.
19. H. Li, U. Simon, Quantization of curvature for compact surfcaes in S^n, *Math. Z.* **245** (2003) 201–216.
20. H. Li and L. Vrancken, New examples of Willmore surfaces in S^n, *Ann. Global Anal. Geom.* **23** (2003) 205–225.
21. H. Li and G.X. Wei, Classification of Lagrangian Willmore submanifolds of the nearly Kaehler 6-sphere $S^6(1)$ with constant scalar curvature, *Glasgow Math. J.* **48** (2006) 53–64.
22. B. Palmer, Stability of the wulff shape, *Proc. Amer. Math. soci.* **126** (12) (1998) 3661–3667.
23. U. Pinkall, Hopf tori in S^3, *Invent. Math.* **81** (1985) 379–386.
24. F.J. Pedit and T.J. Willmore, Conformal geometry, *Atti Sem. Mat. Fis. Univ. Modena* **XXXVI** (1988) 237–245.
25. R. Reilly, Variation properties of functions of the mean curvatures for hypersurfaces in space forms, *J. Diff. Geom.* **8** (1973) 465–477.
26. R. Reilly, Applications of the Hessian operator in a Riemannian manifold, *Indiana Univ. math. Journal* **26** (1977) 459–472.
27. J. Simons, Minimal varieties in Riemmannian manifolds, *Ann. Math.*, 2nd Ser., **88** (1) (1968) 62.
28. C.P. Wang , Moebius geometry of submanifolds in S^n, *Manuscripta Math.* **96** (1998) 517–534.
29. J. Weiner, On a problem of Chen, Willmore, et., *Indiana Univ. Math. J.* **27** (1978) 19–35.
30. T.J. Willmore, Notes on embedded surfaces, *Ann. Stiint. Univ. Al. Cuza*, Iasi, Sect. I a Mat.(N.S.) 11B (1965) 493–496.