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r-Minimal submanifolds in space forms

Linfen Cao · Haizhong Li

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Abstract Let $x: M \to R^{n+p}(c)$ be an n-dimensional compact, possibly with boundary, submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. Assume that r is even and $r \in \{0, 1, \ldots, n-1\}$, in this paper we introduce rth mean curvature function S_r and $(r+1)$-th mean curvature vector field $\mathbf{S}_{r+1}$. We call M to be an r-minimal submanifold if $\mathbf{S}_{r+1} \equiv 0$ on M, we note that the concept of 0-minimal submanifold is the concept of minimal submanifold. In this paper, we define a functional $J_r(x) = \int_M F_r(S_0, S_2, \ldots, S_r)dv$ of $x: M \to R^{n+p}(c)$, by calculation of the first variational formula of J_r we show that x is a critical point of J_r if and only if x is r-minimal. Besides, we give many examples of r-minimal submanifolds in space forms. We calculate the second variational formula of J_r and prove that there exists no compact without boundary stable r-minimal submanifold with $S_r > 0$ in the unit sphere S^{n+p}. When $r = 0$, noting $S_0 = 1$, our result reduces to Simons' result: there exists no compact without boundary stable minimal submanifold in the unit sphere S^{n+p}.

Keywords rth Mean curvature function · $(r+1)$th Mean curvature vector field · L_r operator · r-Minimal submanifold · Stability

Mathematics Subject Classification (2000) Primary 53C40 · Secondary 53C42

1 Introduction

Let $x: M \to R^{n+p}(c)$ be an n-dimensional compact, possibly with boundary, submanifold of an $(n+p)$-dimensional space form $R^{n+p}(c)$ of constant sectional curvature c,

L. Cao · H. Li (✉)
Department of Mathematical Sciences, Tsinghua University,
100084 Beijing, People's Republic of China
e-mail: hli@math.tsinghua.edu.cn

L. Cao
e-mail: caolf04@mails.tsinghua.edu.cn

where $R^{n+p}(c)$ is Euclidean space R^{n+p} when $c=0$, $R^{n+p}(c)$ is a unit sphere S^{n+p} when $c=1$, and $R^{n+p}(c)$ is a hyperbolic space H^{n+p} when $c=-1$. We choose an orthonormal frame field $\{e_1,\ldots,e_n,e_{n+1},\ldots,e_{n+p}\}$ along M such that $\{e_i\}_{i=1}^n$ are tangent to M and $\{e_\alpha\}_{\alpha=n+1}^{n+p}$ are normal to M, their dual frame are $\{\theta_i\}_{i=1}^n$ and $\{\theta_\alpha\}_{\alpha=n+1}^{n+p}$. If h_{ij}^α denote the components of the second fundamental form of the immersion $x\colon M^n\to R^{n+p}(c)$, we write

$$B_{ij}=\sum_{\alpha=n+1}^{n+p}h_{ij}^\alpha e_\alpha. \tag{1.1}$$

If $i_1,\ldots,i_r$ and $j_1,\ldots,j_r$ are integers between 1 and n, then generalized Kronecker symbols $\delta_{j_1\cdots j_r}^{i_1\cdots i_r}$ is $+1$ or -1 according as the i's are distinct and the j's are an even or odd permutation of the $i's$, and is 0 in all other cases. We define the following $(0,2)$-tensor T_r for $r\in\{1,\ldots,n-1\}$, (see [11, 22]):

If r is even, we set

$$T_{rj}^{\ i}=\frac{1}{r!}\sum_{\substack{i_1\cdots i_r\\ j_1\cdots j_r}}\delta_{j_1\cdots j_r j}^{i_1\cdots i_r i}\langle B_{i_1j_1},B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}},B_{i_rj_r}\rangle, \tag{1.2}$$

we also set for a fixed index α, $n+1\le\alpha\le n+p$

$$T_{r-1\ ij}^{\ \ \alpha}=\frac{1}{(r-1)!}\sum_{\substack{i_1\cdots i_{r-1}\\ j_1\cdots j_{r-1}}}\delta_{j_1\cdots j_{r-1}j}^{i_1\cdots i_{r-1}i}\langle B_{i_1j_1},B_{i_2j_2}\rangle\cdots\langle B_{i_{r-3}j_{r-3}},B_{i_{r-2}j_{r-2}}\rangle h_{i_{r-1}j_{r-1}}^{\ \ \alpha} \tag{1.3}$$

Then for any even integer $r\in\{0,1,\ldots,n-1\}$, we introduce rth mean curvature function S_r and $(r+1)$th mean curvature vector field $\mathbf{S}_{r+1}$as follows:

$$\begin{aligned}S_r&=\frac{1}{r!}\sum_{\substack{i_1\cdots i_r\\ j_1\cdots j_r}}\delta_{j_1\cdots j_r}^{i_1\cdots i_r}\langle B_{i_1j_1},B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}},B_{i_rj_r}\rangle\\&=\frac{1}{r}\sum_{i,j,\alpha}T_{r-1\ ij}^{\ \ \alpha}h_{ij}^\alpha,\end{aligned} \tag{1.4}$$

$$\begin{aligned}\mathbf{S}_{r+1}&=\frac{1}{(r+1)!}\sum_{\substack{i_1\cdots i_{r+1}\\ j_1\cdots j_{r+1}}}\delta_{j_1\cdots j_{r+1}}^{i_1\cdots i_{r+1}}\langle B_{i_1j_1},B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}},B_{i_rj_r}\rangle B_{i_{r+1}j_{r+1}}\\&=\frac{1}{r+1}\sum_{i,j,\alpha}T_{rj}^{\ i}h_{ij}^\alpha e_\alpha,\end{aligned} \tag{1.5}$$

$$S_r=\binom{n}{r}H_r,\quad \mathbf{S}_{r+1}=\binom{n}{r+1}\mathbf{H}_{r+1}, \tag{1.6}$$

where $\binom{n}{r}$ being the binomial coefficient, $\delta_{j_1\cdots j_r}^{i_1\cdots i_r}$ is the generalized Kronecker symbol. By convention, we put $H_0=S_0=1$.

Motivated by the works of Barbosa–Colares in [5] about hypersurfaces in space forms, we introduce a curvature integral J_r for an n-dimensional compact submanifold

M in $R^{n+p}(c)$, r is even and $r \in \{0, 1, \dots, n-1\}$

$$J_r = \int_M F_r(S_0, S_2, \dots, S_r) dv$$

where the function F_r are defined inductively by

$$\begin{cases} F_0 = 1 \\ F_r = S_r + \frac{(n-r+1)c}{r-1} F_{r-2}, \quad \text{for } 2 \le r \le n-1. \end{cases}$$

Now we give the variational formula of J_r. In this paper, we consider only deformations which leaves ∂M strongly fixed. Of course, if M is compact and ∂M is empty, then there is no restriction on the deformation.

Theorem 1.1 (the first variational formula) *Let M be an n-dimensional compact, possibly with boundary, submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. Assume that r is even and $r \in \{0, 1, \dots n-1\}$, then we have*

$$J_r{}'(t) = -(r+1) \int_{M_t} \langle \mathbf{S}_{r+1}, V \rangle dv_{g_t}$$

where V is the variational vector field.

Definition 1.1 Let M be an n-dimensional submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. Assume that r is even and $r \in \{0, 1, \dots, n-1\}$, we call x to be r-minimal if its $(r+1)$-th mean curvature vector $\mathbf{S}_{r+1}$ vanishes on M.

Definition 1.2 Assume that r is even and $r \in \{0, 1, \dots, n-1\}$, and let M be an n-dimensional compact r-minimal submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. If $J_r''(0) \ge 0$ for arbitrary variations, we call M to be stable.

In this paper we get the following generalized Takahashi theorem:

Theorem 1.2 *Let $x : M \to R^{n+p}(c)$ be an n-dimensional submanifold of an $(n+p)$-dimensional space form $R^{n+p}(c)$. For any even integer $r \in \{0, 1, \dots, n-1\}$, then M is r-minimal if and only if*

$$L_r x = -(n-r) c S_r x,$$

where the operator L_r is defined in Sect. 4.

Remark 1.1 From Definition 1.1, we know that concept of 0-minimal submanifolds is the concept of minimal submanifolds. When $r = 0$, $L_0 = \triangle$ is the Laplacian of M, our Theorem 1.2 reduces to Takahashi's well-known result about minimal submanifolds (see [25]).

In [24], Simons gave an instability result for minimal submanifolds in the sphere as follows

Theorem 1.3 (Simons [24]) *There exists no compact without boundary stable minimal submanifold in the unit sphere S^{n+p}.*

In this paper, we generalize Simons' result to r-minimal submanifolds. In fact, we prove

Theorem 1.4 *Assume that r is even and* $r \in \{0, 1, \ldots, n-1\}$. *If* S_r *is positive, then there exists no compact without boundary stable r-minimal submanifold in the unit sphere* S^{n+p}.

Remark 1.2 When $r = 0$, i.e. $S_0 = 1$, it is easy to see that our Theorem 1.4 reduces to Simons' Theorem 1.3. When $r \geq 2$, the following Theorem 1.5 tells us that the geodesic sphere ($S_r \equiv 0$) is stable r-minimal hypersurface, thus the assumption condition $S_r > 0$ in Theorem 1.4 is necessary, thus our Theorem 1.4 is the best possible result in this sense.

Theorem 1.5 *Assume that r is even and* $r \in \{1, \ldots, n-1\}$. *If* $S_r \geq 0$, *then any compact without boundary stable r-minimal hypersurface in unit sphere* S^{n+1} *must be a geodesic sphere.*

This paper consists of seven sections. In Sect. 2, we first review Gauss equation, Codazzi equation and Ricci equation for submanifolds in space forms. In Sect. 3, we give the definitions of r-th mean curvature function S_r, $(r+1)$-th mean curvature vector $\mathbf{S}_{r+1}$ and $(0, 2)$-tensor T_r for submanifolds in space forms, we also give some properties of them. In Sect. 4 we introduce the L_r operator and prove the Theorem 1.2 which is the generalized Takahashi theorem. In Sect. 5 we define a curvature integral J_r on submanifolds in space forms and calculate its first variational formula, thus complete the proof of the Theorem 1.1. We also give some examples of r-minimal submanifolds in space forms. In Sect. 6 we calculate the second variational formula for r-minimal submanifolds in space forms. As an application we prove that Theorem 1.4 and Theorem 1.5 in Sect. 7.

2 Preliminaries

In this section, we give some formulas and notations of submanifolds in space forms by using the method of moving frames. Let $x\colon M \to R^{n+p}(c)$ be an n-dimensional submanifold of an $(n+p)$-dimensional space form $R^{n+p}(c)$ of constant sectional curvature c. We will agree on the following index convention:

$$1 \leq i, j, k \cdots \leq n; \quad n+1 \leq \alpha, \beta, \gamma \cdots \leq n+p; \quad 1 \leq A, B, C \cdots \leq n+p.$$

Let $\{e_A\}$ be a local orthonormal basis for $TR^{n+p}(c)$ with dual basis $\{\theta_A\}$ such that when restricted to M, $\{e_i\}$ is a local orthonormal basis for TM and $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of $x\colon M \to R^{n+p}(c)$.

Then we have the structure equations (see [9])

$$dx = \sum_i \theta_i e_i, \tag{2.1}$$

$$de_i = \sum_j \theta_{ij} e_j + \sum_{\alpha, j} h^\alpha_{ij} \theta_j e_\alpha - c\theta_i x \tag{2.2}$$

$$de_\alpha = -\sum_{ij} h^\alpha_{ij} \theta_j e_i + \sum_\beta \theta_{\alpha\beta} e_\beta \tag{2.3}$$

where h^α_{ij} denote the components of the second fundamental form of M.

The Gauss equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})c + \sum_{\alpha}(h^{\alpha}_{ik}h^{\alpha}_{jl} - h^{\alpha}_{il}h^{\alpha}_{jk}) \tag{2.4}$$

$$R_{ik} = (n-1)c\delta_{ik} + n\sum_{\alpha} H^{\alpha}h^{\alpha}_{ik} - \sum_{\alpha,j} h^{\alpha}_{ij}h^{\alpha}_{jk} \tag{2.5}$$

$$R = n(n-1)c + n^2H^2 - S \tag{2.6}$$

where R is the scalar curvature of M and $S = \sum_{\alpha,i,j}(h^{\alpha}_{ij})^2$ is the norm square of the second fundamental form, $\mathbb{H} = \sum_{\alpha} H^{\alpha}e_{\alpha} = \frac{1}{n}\sum_{\alpha}(\sum_{i} h^{\alpha}_{ii})e_{\alpha} = \frac{1}{n}\mathbf{S}_1$ is the mean curvature vector field and $H = |\mathbb{H}|$ is the mean curvature of M.

The Codazzi equations are (see [15, 17, 18, 20])

$$h^{\alpha}_{ijk} = h^{\alpha}_{ikj} \tag{2.7}$$

where the covariant derivative of h^{α}_{ij} is defined by

$$\sum_{k} h^{\alpha}_{ijk}\theta_k = dh^{\alpha}_{ij} + \sum_{k} h^{\alpha}_{kj}\theta_{ki} + \sum_{k} h^{\alpha}_{ik}\theta_{kj} + \sum_{\beta} h^{\beta}_{ij}\theta_{\beta\alpha}. \tag{2.8}$$

If we denote by $R_{\alpha\beta ij}$ the curvature tensor of normal connection $\theta_{\alpha\beta}$ in the normal bundle of $x\colon M \to R^{n+p}(c)$, Then the Ricci equations are

$$R_{\alpha\beta ij} = \sum_{k}(h^{\alpha}_{ik}h^{\beta}_{kj} - h^{\alpha}_{jk}h^{\beta}_{ki}). \tag{2.9}$$

Let f be a smooth function on M, we define the first, the second covariant derivative f_i, f_{ij} and the Laplacian of f as follows

$$df = \sum_{i} f_i\theta_i, \quad \sum_{j} f_{ij}\theta_j = df_i + \sum_{j} f_j\theta_{ji}, \quad \Delta f = \sum_{i} f_{ii} \tag{2.10}$$

3 *r*th Mean curvature function and $(r+1)$th mean curvature vector field for submanifolds in space forms

We will follow notations in the introduction and preliminaries. We recall the definition of the generalized Kronecker symbols. If $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$ are integers between 1 and n, then $\delta^{i_1\ldots i_r}_{j_1\ldots j_r}$ is $+1$ or -1 according as the i's are distinct and the j's are an even or odd permutation of the $i's$, and is 0 in all other cases. In fact, it easy to check that

$$\delta^{i_1\ldots i_r}_{j_1\ldots j_r} = \begin{vmatrix} \delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_r} \\ \vdots & \vdots & \cdots & \vdots \\ \delta^{i_r}_{j_1} & \delta^{i_r}_{j_2} & \cdots & \delta^{i_r}_{j_r} \end{vmatrix} = \delta^{j_1\ldots j_r}_{i_1\ldots i_r}.$$

Let (M, g) be an n-dimensional submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)(c = 1, 0$ or -1, respectively for Euclidean space R^{n+p}, unit sphere S^{n+p} or hyperbolic space H^{n+p}) whose canonical metric will be denoted by $\tilde{g}$ and B be the second fundamental form of M. Suppose $\{e_i\}$ is a local orthonormal basis for TM with dual basis $\{\theta_i\}$ and $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of $x: M^n \to R^{n+p}(c)$, (B_{ij}) is the matrix with respect to the frame $\{e_1, \ldots, e_n\}$ on M. We define the following $(0,2)$-tensor T_r for $r \in \{1, \ldots, n-1\}$, (see [11, 22]):

If r is even, we set

$$T_r = \frac{1}{r!} \sum_{\substack{i_1 \ldots i_r i \\ j_1 \ldots j_r j}} \delta^{i_1 \ldots i_r i}_{j_1 \ldots j_r j} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}}, B_{i_r j_r} \rangle \theta_i \otimes \theta_j$$
$$= \sum_{i,j} T_{rj}^{\ i} \theta_i \otimes \theta_j,$$

that is

$$T_{rj}^{\ i} = \frac{1}{r!} \sum_{\substack{i_1 \ldots i_r \\ j_1 \ldots j_r}} \delta^{i_1 \ldots i_r i}_{j_1 \ldots j_r j} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}}, B_{i_r j_r} \rangle, \tag{3.1}$$

we also set

$$T_{r-1} = \frac{1}{(r-1)!} \sum_{\substack{i_1 \ldots i_{r-1} i \\ j_1 \ldots j_{r-1} j}} \delta^{i_1 \ldots i_{r-1} i}_{j_1 \ldots j_{r-1} j} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-3} j_{r-3}}, B_{i_{r-2} j_{r-2}} \rangle B_{i_{r-1} j_{r-1}} \theta_i \otimes \theta_j$$
$$= \sum_{i,j,\alpha} T_{r-1\ ij}^{\ \alpha} \theta_i \otimes \theta_j e_\alpha,$$

that is, for a fixed index α, $n+1 \leq \alpha \leq n+p$

$$T_{r-1\ ij}^{\ \alpha} = \frac{1}{(r-1)!} \sum_{\substack{i_1 \ldots i_{r-1} \\ j_1 \ldots j_{r-1}}} \delta^{i_1 \ldots i_{r-1} i}_{j_1 \ldots j_{r-1} j} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-3} j_{r-3}}, B_{i_{r-2} j_{r-2}} \rangle h^{\ \alpha}_{i_{r-1} j_{r-1}}. \tag{3.2}$$

Then for any even integer $r \in \{0, 1, \ldots, n-1\}$, we introduce rth mean curvature function S_r and $(r+1)$th mean curvature vector field $\mathbf{S}_{r+1}$ as follows:

$$S_r = \frac{1}{r!} \sum_{\substack{i_1 \ldots i_r \\ j_1 \ldots j_r}} \delta^{i_1 \ldots i_r}_{j_1 \ldots j_r} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}}, B_{i_r j_r} \rangle$$
$$= \frac{1}{r} \sum_{i,j,\alpha} T_{r-1\ ij}^{\ \alpha} h^\alpha_{ij}, \tag{3.3}$$

$$\mathbf{S}_{r+1} = \frac{1}{(r+1)!} \sum_{\substack{i_1 \ldots i_{r+1} \\ j_1 \ldots j_{r+1}}} \delta^{i_1 \ldots i_{r+1}}_{j_1 \ldots j_{r+1}} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}}, B_{i_r j_r} \rangle B_{i_{r+1} j_{r+1}}$$
$$= \frac{1}{r+1} \sum_{i,j,\alpha} T_{rj}^{\ i} h^\alpha_{ij} e_\alpha, \tag{3.4}$$

$$S_r = \binom{n}{r} H_r, \quad \mathbf{S}_{r+1} = \binom{n}{r+1} \mathbf{H}_{r+1}, \tag{3.5}$$

where $\binom{n}{r}$ being the binomial coefficient, $\delta^{i_1\ldots i_r}_{j_1\ldots j_r}$ is the generalized Kronecker symbol. By convention, we put $H_0 = S_0 = 1$.

By convention $T_0 =$ identity. When the codimension $p = 1$, we have the unified formula for the tensors T_r. In this case, we denote the unit normal vector to M (determined by the orientation on M) by e_{n+1}, if r is odd we replace T_r by the tensor $T_r = \tilde{g}(T_r(\cdot,\cdot), e_{n+1})$. Write $h^{n+1}_{ij} = h_{ij}$, then we have

$$\begin{aligned} T_r &= \frac{1}{r!} \sum_{\substack{i_1\ldots i_r i\\ j_1\ldots j_r j}} \delta^{i_1\ldots i_r i}_{j_1\ldots j_r j} h_{i_1j_1}\cdots h_{i_rj_r}\theta_i\otimes\theta_j \\ &= \sum_{i,j} T_{r\,j}^{\;i}\theta_i\otimes\theta_j. \end{aligned}$$

The following properties of T_r are fundamental

Lemma 3.1 *Let M be an n-dimensional submanifold in $R^{n+p}(c)$. Let r be even and $r \in \{0, 1, \ldots, n-1\}$. Then T_r is a symmetric and divergence-free $(0, 2)$-tensor, that is*

$$T_{r\,j}^{\;i} = T_{r\,i}^{\;j}, \quad \sum_{i=1}^{n} T_{r\,j,\,i}^{\;i} = 0. \tag{3.6}$$

Proof By (3.1), we have the following calculation:

$$\begin{aligned} T_{r\,j}^{\;i} &= \frac{1}{r!} \sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r}} \delta^{i_1\ldots i_r i}_{j_1\ldots j_r j}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}}, B_{i_rj_r}\rangle \\ &= \frac{1}{r!} \sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r}} \delta^{j_1\ldots j_r j}_{i_1\ldots i_r i}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}}, B_{i_rj_r}\rangle \\ &= \frac{1}{r!} \sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r}} \delta^{i_1\ldots i_r j}_{j_1\ldots j_r i}\langle B_{j_1i_1}, B_{j_2i_2}\rangle\cdots\langle B_{j_{r-1}i_{r-1}}, B_{j_ri_r}\rangle \\ &= \frac{1}{r!} \sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r}} \delta^{i_1\ldots i_r j}_{j_1\ldots j_r i}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}}, B_{i_rj_r}\rangle \\ &= T_{r\,i}^{\;j}, \end{aligned}$$

$$\begin{aligned} \mathrm{div} T_r &= \sum_{i=1}^{n} T_{r\,j,\,i}^{\;i} \\ &= \frac{1}{r!} \sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ i}} \delta^{i_1\ldots i_r i}_{j_1\ldots j_r j}\{\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}}, B_{i_rj_r}\rangle\}_{,i} \\ &= \frac{1}{(r-1)!} \sum_{\substack{i_1,\ldots i_r\\ j_1\ldots j_r\\ i,\alpha}} \delta^{i_1\ldots i_r i}_{j_1\ldots j_r j}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-3}j_{r-3}}, B_{i_{r-2}j_{r-2}}\rangle h^{\alpha}_{i_{r-1}j_{r-1}} h^{\alpha}_{i_rj_ri} \end{aligned}$$

$$\overset{(1)}{=} \frac{1}{(r-1)!} \sum_{\substack{i_1...i_r \\ j_1...j_r \\ i,\alpha}} \delta^{i_1...i_r i}_{j_1...j_r j} \langle B_{i_1 j_1}, B_{i_2 j_2}\rangle \cdots \langle B_{i_{r-3} j_{r-3}}, B_{i_{r-2} j_{r-2}}\rangle h^{\alpha}_{i_{r-1} j_{r-1}} h^{\alpha}_{i_r i j_r}$$

$$\overset{(2)}{=} 0$$

The equality (1) follows from the Codazzi equation (2.7), the equality (2) use the fact that the pair (i_r, i) is anti-symmetric in $\delta^{i_1...i_r i}_{j_1...j_r j}$ and is symmetric in the $h^{\alpha}_{i_r i j_r}$. □

Lemma 3.2 *Let M be an n-dimensional submanifold in $R^{n+p}(c)$. Assume that r is odd and $r \in \{1, \ldots, n-1\}$. Then T_r is a symmetric and divergence-free normal-vectored value $(0, 2)$-tensor. That is, for a fixed index α, $n+1 \leq \alpha \leq n+p$, we have*

$$T_{r\,ij}^{\ \alpha} = T_{r\,ji}^{\ \alpha}; \quad \sum_{j=1}^{n} T_{r\,ij,j}^{\ \alpha} = 0. \tag{3.7}$$

Proof

$$\begin{aligned}
\sum_{j=1}^{n} T_{r\,ij,j}^{\ \alpha} &= \frac{1}{r!} \sum_{\substack{i_1...i_r \\ j_1...j_r \\ j}} \delta^{i_1...i_r i}_{j_1...j_r j} \{\langle B_{i_1 j_1}, B_{i_2 j_2}\rangle \cdots \langle B_{i_{r-2} j_{r-2}}, B_{i_{r-1} j_{r-1}}\rangle h^{\alpha}_{i_r j_r}\}_{,j} \\
&= \frac{1}{r!} \sum_{\substack{i_1...i_r \\ j_1...j_r \\ j}} \delta^{i_1...i_r i}_{j_1...j_r j} \{(\langle B_{i_1 j_1}, B_{i_2 j_2}\rangle \cdots \langle B_{i_{r-2} j_{r-2}}, B_{i_{r-1} j_{r-1}}\rangle)_{,j} h^{\alpha}_{i_r j_r} \\
&\quad + \langle B_{i_1 j_1}, B_{i_2 j_2}\rangle \cdots \langle B_{i_{r-2} j_{r-2}}, B_{i_{r-1} j_{r-1}}\rangle h^{\alpha}_{i_r j_r j}\} \\
&= \frac{1}{r!} \sum_{\substack{i_1,...i_r \\ j_1...j_r \\ j}} \{(r-1) \delta^{i_1...i_{r-1} i_r i}_{j_1...j_{r-1} j_r j} \sum_{\beta} \langle B_{i_1 j_1}, B_{i_2 j_2}\rangle \cdots \langle B_{i_{r-4} j_{r-4}}, B_{i_{r-3} j_{r-3}}\rangle \cdot \\
&\quad h^{\beta}_{i_{r-2} j_{r-2}} h^{\beta}_{i_{r-1} j_{r-1} j} h^{\alpha}_{i_r j_r} + \delta^{i_1...i_r i}_{j_1...j_r j} \langle B_{i_1 j_1}, B_{i_2 j_2}\rangle \cdots \langle B_{i_{r-2} j_{r-2}}, B_{i_{r-1} j_{r-1}}\rangle h^{\alpha}_{i_r j_r j}\} \\
&= 0
\end{aligned}$$

The last equality holds since in the first term the pair (j_{r-1}, j) is symmetric in $h^{\ \alpha}_{i_{r-1} j_{r-1} j}$ according to the Codazzi equation (2.7) and is anti-symmetric in the Kronecker symbols $\delta^{i_1...i_{r-1} i_r i}_{j_1...j_{r-1} j_r j}$, thus the sums over $\{j_{r-1}, j\}$ vanish, the second term vanishes in similar reason with respect to pair (j_r, j).

By use of similar skills as in the proof of Lemma 3.1, it is easy to verify the symmetry of $T_{r\,ij}^{\ \alpha}$ for a fixed index α. □

Now we will give the relations between the rth mean curvature functions, rth mean curvature vector fields and the tensors T_r, some of these relations are given in [11]. When the codimension is 1, the related results can be found in [1–3, 5–7, 10, 21, 23].

Lemma 3.3 *For any integer $r \in \{0, 1, \ldots, n-1\}$, when r is even, we have*

$$\mathrm{trace}(T_r) = (n-r)S_r \tag{3.8}$$

and when r is odd, for each $n+1 \leq \alpha \leq n+p$, we have

$$\operatorname{trace}(T_r{}^{\alpha}_{ij}) = \frac{n-r}{r}\sum_{i,j} T_{r-1}{}^{i}_{j} h^{\alpha}_{ij}. \tag{3.9}$$

Proof We have by use of (3.1), (3.3) and (3.5)

$$\begin{aligned}
\operatorname{trace}(T_r) &= \sum_i T_r{}^i_i \\
&= \frac{1}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ i}} \delta^{i_1\ldots i_r i}_{j_1\ldots j_r i}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}}, B_{i_rj_r}\rangle \\
&= \frac{1}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ i\notin\{i_1,\ldots,i_r\}}} \delta^{i_1\ldots i_r i}_{j_1\ldots j_r i}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}}, B_{i_rj_r}\rangle \\
&= \frac{n-r}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r}} \delta^{i_1\ldots i_r}_{j_1\ldots j_r}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}}, B_{i_rj_r}\rangle \\
&= (n-r)S_r,
\end{aligned}$$

$$\begin{aligned}
\operatorname{trace}(T_r{}^{\alpha}_{ij}) &= \sum_i T_r{}^{\alpha}_{ii} \\
&= \frac{1}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ i}} \delta^{i_1\ldots i_r i}_{j_1\ldots j_r i}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-2}j_{r-2}}, B_{i_{r-1}j_{r-1}}\rangle h^{\alpha}_{i_rj_r} \\
&= \frac{1}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ i\notin\{i_1,\ldots,i_r\}}} \delta^{i_1\ldots i_r i}_{j_1\ldots j_r i}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-2}j_{r-2}}, B_{i_{r-1}j_{r-1}}\rangle h^{\alpha}_{i_rj_r} \\
&= \frac{n-r}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r}} \delta^{i_1\ldots i_r}_{j_1\ldots j_r}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-2}j_{r-2}}, B_{i_{r-1}j_{r-1}}\rangle h^{\alpha}_{i_rj_r} \\
&= \frac{n-r}{r}\sum_{i,j} T_{r-1}{}^{i}_{j}\, h^{\alpha}_{ij}.
\end{aligned}$$

□

Lemma 3.4 *For any integer $r \in \{1,\ldots,n-1\}$, when r is even, we have*

$$T_r{}^i_j = S_r\delta^i_j - \sum_{k,\alpha} T_{r-1}{}^{\alpha}_{kj}\, h^{\alpha}_{ki} \tag{3.10}$$

Proof We have the following calculations

$$\begin{aligned}
T_r{}^i_j &= \frac{1}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r}} \delta^{i_1\ldots i_r i}_{j_1\ldots j_r j}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}}, B_{i_rj_r}\rangle \\
&= \frac{1}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r}} \left\{\delta^i_j\delta^{i_1\ldots i_r}_{j_1\ldots j_r} - \delta^i_{j_r}\delta^{i_1\ldots i_{r-1}i_r}_{j_1\ldots j_{r-1}j} + \delta^i_{j_{r-1}}\delta^{i_1\ldots i_{r-2}i_{r-1}i_r}_{j_1\ldots j_{r-2}j_r\ \ j} + \cdots\right\} \\
&\quad \cdot\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}}, B_{i_rj_r}\rangle
\end{aligned}$$

$$= S_r\delta^i_j + \frac{1}{r!}\left\{ - \sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ \alpha}} \delta^{i_1\ldots i_{r-1} i_r}_{j_1\ldots j_{r-1}\, j}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-3}j_{r-3}}, B_{i_{r-2}j_{r-2}}\rangle h^{\ \alpha}_{i_{r-1}j_{r-1}}\ h^\alpha_{i_ri}\right.$$

$$\left. + \sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ \alpha}} \delta^{i_1\ldots i_{r-2} i_{r-1} i_r}_{j_1\ldots j_{r-2} j_r\ \ j}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-3}j_{r-3}}, B_{i_{r-2}j_{r-2}}\rangle h^{\ \alpha}_{i_{r-1}i}\ h^{\ \alpha}_{i_rj_r} + \cdots \right\}$$

$$= S_r\delta^i_j - \frac{r}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ \alpha}} \delta^{i_1\ldots i_{r-1} i_r}_{j_1\ldots j_{r-1}\, j}\langle B_{i_1j_1}, B_{i_2j_2}\rangle\cdots\langle B_{i_{r-3}j_{r-3}}, B_{i_{r-2}j_{r-2}}\rangle h^{\ \alpha}_{i_{r-1}j_{r-1}}\ h^\alpha_{i_ri}$$

$$= S_r\delta^i_j - \sum_{i_r,\alpha} T_{r-1}{}^{\alpha}_{i_rj}\ h^\alpha_{i_ri}$$

$$= S_r\delta^i_j - \sum_{k,\alpha} T_{r-1}{}^{\alpha}_{kj}\ h^\alpha_{ki}. \qquad \square$$

4 L_r operator and proof of Theorem 1.2

Let $x: M \to R^{n+1}(c)$ be an n-dimensional compact without boundary hypersurface in $R^{n+1}(c)$, associated to each Newton transformation T_r of an immersion x, a second order differential operator L_r defined by (see [8, 23])

$$L_rf = \sum_{i,j} T_{r}{}^{i}_{j}f_{ij}$$

where f is any smooth function on M.

In Sect. 3, the definitions of operator T_r for submanifolds in the space form were given when r is even and r is odd, respectively, so we have the corresponding operator L_r.

- If r is even,

$$L_r\colon C^\infty(M) \to C^\infty(M)$$
$$f \mapsto \sum_{i,j} T_{r}{}^{i}_{j}f_{ij}$$

- If r is odd,

$$L_r\colon C^\infty(M) \to \Gamma(T^\perp M)$$
$$f \mapsto \sum_{i,j,\alpha} T_{r}{}^{\alpha}_{ij}f_{ij}e_\alpha$$

where $\Gamma(T^\perp M)$ denotes the section of the normal bundle of M.

When r is odd, denote $L^*_r\colon \Gamma(T^\perp M) \to C^\infty(M)$ the adjoint operator of L_r above, i.e.

$$\int_M fL^*_r(\xi)dv = \int_M L_r(f)\cdot\xi\, dv \tag{4.1}$$

where

$$L_r^*\xi = \sum_{i,j,\alpha} T_r{}_{ij}^{\alpha}\, \xi_{\alpha,ij}, \quad \xi = \sum_{\alpha} \xi_\alpha e_\alpha \in \Gamma(T^\perp M) \tag{4.2}$$

$\xi_{\alpha,i}$ and $\xi_{\alpha,ij}$ are the first and the second covariant derivative of ξ defined by, respectively

$$\sum_i \xi_{\alpha,i}\theta_i = d\xi_\alpha + \sum_\beta \xi_\beta \theta_{\beta\alpha}$$

$$\sum_j \xi_{\alpha,ij}\theta_j = d\xi_{\alpha,i} + \sum_\beta \xi_{\beta,i}\theta_{\beta\alpha} + \sum_j \xi_{\alpha,j}\theta_{ji}$$

Choosing $f = 1$ in (4.1), we get the integral formula

$$\int_M L_r^*(\xi)dv = 0 \tag{4.3}$$

Remark 4.1 By the Lemma 3.1, when r is even, $L_r(f) = \mathrm{div}(T_r\nabla f)$.

In order to prove Theorem 1.2, we first prove the following important lemma:

Lemma 4.1 *Let $x : M \to R^{n+p}(c)$ be an n-dimensional submanifold of an $(n+p)$-dimensional space form $R^{n+p}(c)$. Suppose $\{e_i\}$ is a local orthonormal basis for TM and $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of $x: M^n \to R^{n+p}(c)$. For any integer $r \in \{0, 1, \ldots, n-1\}$, when r is even, we have*

$$L_r(x) = (r+1)\mathbf{S}_{r+1} - c\cdot(n-r)S_r x \tag{4.4}$$

$$L_r(e_\alpha) = -\sum_{i,j,k} T_{rj}^{\,i} h_{ik,j}^{\alpha} e_k - \sum_{i,j,k,\beta} T_{rj}^{\,i} h_{ik}^{\alpha} h_{kj}^{\beta} e_\beta + c\cdot\sum_{i,j} T_{rj}^{\,i} h_{ij}^{\alpha} x. \tag{4.5}$$

Remark 4.2 When $p = 1$, above Lemma 4.1 was proved by Reilly [21] (also see Rosenberg [23], Barbosa-Colares [5] or Alencar-Rosenberg-Santos [4]).

Proof For $c = 0$, let a be a fixed vector in R^{n+p}; when $c > 0$, let a be a fixed vector in R^{n+p+1}; when $c < 0$, let a be a fixed vector in R_1^{n+p+1}. We define the following height functions in the a direction on M,

$$f = \langle x, a\rangle \tag{4.6}$$

$$g_\alpha = \langle e_\alpha, a\rangle \tag{4.7}$$

for a fixed normal vector e_α. From the Definition (2.10) of f_i, we have by use of the structure equation (2.1)

$$f_i = \langle e_i, a\rangle. \tag{4.8}$$

From the Definition (2.10) of f_{ij}, we have by use of the structure equation (2.2)

$$f_{ij} = \sum_\alpha h_{ij}^\alpha \langle e_\alpha, a\rangle - c\delta_{ij}\langle x, a\rangle \tag{4.9}$$

From arbitrary property of a in (4.9), we have

$$x_{ij} = \sum_{\alpha} h^{\alpha}_{ij} e_{\alpha} - c\delta_{ij} x \tag{4.10}$$

Define the first derivative $g_{\alpha,i}$ of g_{α} by

$$\sum_{i} g_{\alpha,i}\theta_i = dg_{\alpha} + \sum_{\beta} g_{\beta}\theta_{\beta\alpha}. \tag{4.11}$$

We have by use of (2.3)

$$g_{\alpha,i} = -\sum_{j} h^{\alpha}_{ij}\langle e_j, a\rangle. \tag{4.12}$$

Taking covariant derivatives on both sides of (4.12) in the e_j direction, we have by use of (4.9)

$$g_{\alpha,ij} = -\sum_{k} h^{\alpha}_{ikj}\langle e_k, a\rangle - \sum_{k,\beta} h^{\alpha}_{ik}h^{\beta}_{kj}\langle e_{\beta}, a\rangle + c\cdot h^{\alpha}_{ij}\langle x, a\rangle. \tag{4.13}$$

where the second derivative $g_{\alpha,ij}$ of g_{α} is defined by

$$\sum_{j} g_{\alpha,ij}\theta_j = dg_{\alpha,i} + \sum_{j} g_{\alpha,j}\theta_{ji} + \sum_{\beta} g_{\beta,i}\theta_{\beta\alpha}. \tag{4.14}$$

From arbitrary property of a in (4.13), we have

$$e_{\alpha,ij} = -\sum_{k} h^{\alpha}_{ikj}e_k - \sum_{k,\beta} h^{\alpha}_{ik}h^{\beta}_{kj}e_{\beta} + c\cdot h^{\alpha}_{ij}x, \tag{4.15}$$

where the covariant derivative h^{α}_{ijk} of the second fundamental form h^{α}_{ij} is defined by (2.8). So we have by use of (3.4) and Lemma 3.3

$$\begin{aligned} L_r(x) &= \sum_{i,j} T_{rj}^{\,i} x_{,ij} = \sum_{i,j} T_{rj}^{\,i}(h^{\alpha}_{ij}e_{\alpha} - c\delta_{ij}x) \\ &= (r+1)\mathbf{S}_{r+1} - c\cdot(n-r)S_r x, \end{aligned}$$

$$\begin{aligned} L_r(e_{\alpha}) &= \sum_{i,j} T_{rj}^{\,i} e_{\alpha,ij} = \sum_{i,j} T_{rj}^{\,i}\left(-\sum_{k} h^{\alpha}_{ikj}e_k - \sum_{k,\beta} h^{\alpha}_{ik}h^{\beta}_{kj}e_{\beta} + c\cdot h^{\alpha}_{ij}x\right) \\ &= -\sum_{i,j,k} T_{rj}^{\,i}h^{\alpha}_{ikj}e_k - \sum_{i,j,k,\beta} T_{rj}^{\,i}h^{\alpha}_{ik}h^{\beta}_{kj}e_{\beta} + c\cdot\sum_{i,j} T_{rj}^{\,i}h^{\alpha}_{ij}x. \end{aligned}$$

□

Proof of Theorem 1.2 We complete the proof of Theorem 1.2 by use of (4.4) and the definition of r-minimal submanifold.

Remark 4.3 As a direct application of (4.4), we can easily get the following integral formulas of Minkowski-Hsiung's type.

Corollary 4.1 *M be an n-dimensional compact without boundary submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)(c \neq 0)$, then we have for any even integer $r \in \{0, 1, \ldots, n-1\}$*

$$\int_M \left\{(r+1)\langle\mathbf{S}_{r+1}, a\rangle - c\cdot(n-r)S_r\langle x, a\rangle\right\} dv = 0, \tag{4.16}$$

where a is any fixed vector in R^{n+p+1} *in case* $c > 0$ *and in* R_1^{n+p+1} *in case* $c < 0$.

5 The first variational formula, proof of Theorem 1.1 and some examples of *r*-minimal submanifolds

Let $x_0 : M \to R^{n+p}(c)$ be an n-dimensional compact, possibly with boundary, submanifold of an $(n+p)$-dimensional space form $R^{n+p}(c)$. For each even integer $r \in \{0, 1, \ldots, n-1\}$ we define

$$J_r = \int_M F_r(S_0, S_2, \ldots, S_r) dv_g \tag{5.1}$$

where function F_r are defined inductively by

$$\begin{cases} F_0 = 1 \\ F_r = S_r + \frac{(n-r+1)c}{r-1} F_{r-2}, \quad \text{for } \ 2 \le r \le n-1. \end{cases} \tag{5.2}$$

In order to calculate the Euler–Lagrange equation of $J_r(t)$, we first calculate the derivative of $S_r(t)$ with respect to t.

Proposition 5.1 *Let M be an n-dimensional compact, possibly with boundary, submanifold in an* $(n+p)$*-dimensional space form* $R^{n+p}(c)$*. Assume that r is even and* $r \in \{0, 1, \ldots, n-1\}$*, we have*

$$\frac{\partial S_r}{\partial t} = \sum_\alpha L^*_{r-1}(V_\alpha e_\alpha) + \sum_i S_{r,\,i} V_i + c \cdot (n-r+1)\mathbf{S}_{r-1}, V\rangle$$
$$- (r+1)\mathbf{S}_{r+1}, V\rangle + S_r \mathbf{S}_1, V\rangle, \tag{5.3}$$
$$\frac{\partial}{\partial t} \int_{M_t} S_r dv_{g_t} = \int_{M_t} \langle -(r+1\mathbf{S}_{r+1} + c \cdot (n-r+1\mathbf{S}_{r-1}, V\rangle dv_{g_t} \tag{5.4}$$

where $V = \sum_i V_i e_i + \sum_\alpha V_\alpha e_\alpha$ *is the variational vector field of* $x_0 : M^n \to R^{n+p}(c)$, $S_{r,i}$ *is the first covariant derivative of* S_r *in the* e_i *direction.*

Remark 5.1 We note that (5.4) was proved by Li by a different method [16] in different settings. When $p = 1$, for any integer r, (5.3) and (5.4) were got by Reilly [21] (also see Rosenberg [23], Barbosa–Colares [5] etc.)

Let $x : M \times R \to R^{n+p}(c)$ be a smooth variation of x_0 such that $x(\cdot, t) = x_0$. Along $x : M \times R \to R^{n+p}(c)$, we choose a local orthonormal basis $\{e_A\}$ for $TR^{n+p}(c)$ with dual basis $\{\omega_A\}$, such that $\{e_i(\cdot, t)\}$ forms a local orthonormal basis for $x_t : M \times \{t\} \to R^{n+p}(c)$. Since $T^*(M \times R) = T^*M \oplus T^*R$, the pullback of $\{\omega_A\}$ and $\{\omega_{AB}\}$ on $R^{n+p}(c)$ through $x : M \times R \to R^{n+p}(c)$ have the decomposition

$$x^*\omega_\alpha = V_\alpha dt, \quad x^*\omega_i = \theta_i + V_i dt, \tag{5.5}$$

$$x^*\omega_{ij} = \theta_{ij} + a_{ij}dt, \quad x^*\omega_{i\alpha} = \theta_{i\alpha} + a_{i\alpha}dt, \quad x^*\omega_{\alpha\beta} = \theta_{\alpha\beta} + a_{\alpha\beta}dt, \tag{5.6}$$

where $\{V_i, V_\alpha, a_{ij}, a_{i\alpha}, a_{\alpha\beta}\}$ are local functions on $M \times R$ with $a_{ij} = -a_{ji}$, $a_{\alpha\beta} = -a_{\beta\alpha}$ and

$$V = \frac{d}{dt}\Big|_{t=0} x_t = \sum_i V_i dx_0(e_i) + \sum_\alpha V_\alpha e_\alpha, \tag{5.7}$$

is the variational vector field of x_t: $M \to R^{n+p}(c)$. We note that the one forms $\{\theta_i, \theta_{ij}, \theta_{i\alpha}, \theta_{\alpha\beta}\}$ are defined on $M \times \{t\}$, for $t = 0$, they reduce to the forms with the same notation on M.

We denote by d_M the differential operator on T^*M, then we have $d = d_M + dt\frac{\partial}{\partial t}$ on $T^*(M \times R)$.

Lemma 5.1 *Under the above notations, we have*

$$\frac{\partial \theta_i}{\partial t} = \sum_j (V_{i,j} + a_{ij})\theta_j - \sum_{j,\alpha} h^\alpha_{ij} V_\alpha \theta_j, \tag{5.8}$$

$$a_{i\alpha} = V_{\alpha,i} + \sum_j h^\alpha_{ij} V_j, \tag{5.9}$$

$$\frac{\partial \theta_{i\alpha}}{\partial t} = \sum_j \Big(a_{i\alpha,j} + \sum_k a_{ik} h^\alpha_{jk} - \sum_\beta a_{\beta\alpha} h^\beta_{ij} + c \cdot \delta_{ij} V_\alpha\Big)\theta_j, \tag{5.10}$$

where h^α_{ij} and the covariant derivatives $V_{i,j}$, $V_{\alpha,i}$ and $a_{i\alpha,j}$ are defined on $M \times \{t\}$ by

$$\theta_{i\alpha} = \sum_j h^\alpha_{ij}\theta_j, \tag{5.11}$$

$$\sum_j V_{i,j}\theta_j = d_M V_i + \sum_j V_j \theta_{ji}, \tag{5.12}$$

$$\sum_i V_{\alpha,i}\theta_i = d_M V_\alpha + \sum_\beta V_\beta \theta_{\beta\alpha}, \tag{5.13}$$

$$\sum_j a_{i\alpha,j}\theta_j = d_M a_{i\alpha} + \sum_j a_{j\alpha}\theta_{ji} + \sum_\beta a_{i\beta}\theta_{\beta\alpha}. \tag{5.14}$$

Proof These are direct calculations. In fact, substituting (5.5) and (5.6) into the following equations, respectively

$$d(x^*\omega_i) = x^*(d\omega_i) = x^*\left(\sum_j \omega_{ij} \wedge \omega_j + \sum_\alpha \omega_{i\alpha} \wedge \omega_\alpha\right),$$

$$d(x^*\omega_\alpha) = x^*(d\omega_\alpha) = x^*\left(\sum_j \omega_{\alpha j} \wedge \omega_j + \sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta\right),$$

$$d(x^*\omega_{i\alpha}) = x^*(d\omega_{i\alpha}) = x^*\left(\sum_j \omega_{ij} \wedge \omega_{j\alpha} + \sum_\beta \omega_{i\beta} \wedge \omega_{\beta\alpha} - c \cdot \omega_i \wedge \omega_\alpha\right),$$

and comparing the terms in $T^*M \wedge dt$ for each equation on the both sides, we can get (5.8), (5.9) and (5.10), respectively.

Lemma 5.2

$$\frac{\partial h^\alpha_{ij}}{\partial t} = V_{\alpha,ij} + \sum_k \left(a_{ik}h^\alpha_{kj} + a_{jk}h^\alpha_{ki} + h^\alpha_{ijk}V_k\right) + \sum_\beta a_{\alpha\beta}h^\beta_{ij} + c\cdot\delta_{ij}V_\alpha + \sum_{k,\beta} h^\alpha_{ik}h^\beta_{kj}V_\beta. \tag{5.15}$$

Remark 5.2 When $p = 1$, Lemma 5.2 was proved by Barbosa–Colares [5] (see Lemma 6.1 of [5]). We note that Lemma 5.2 in case $c = 1$ can be found in [12]. Some interesting results about related variational problems for hypersurfaces in space forms can be found in Reilly [21], Rosenberg [23] and Alencar et al. [1]. We also note that Lemma 5.2 was proved by Hu–Li in [15] for any n-dimensional compact submanifolds in an $(n+p)$-dimensional Riemannian manifold N^{n+p}.

Proof Differentiating (5.11) with respect to t and using (5.8), (5.10) we get

$$\frac{\partial h^\alpha_{ij}}{\partial t} = a_{i\alpha,j} + \sum_k a_{ik}h^\alpha_{jk} - \sum_\beta a_{\beta\alpha}h^\beta_{ij} + c\cdot\delta_{ij}V_\alpha - \sum_k\left(h^\alpha_{ik}V_{k,j} + h^\alpha_{ik}a_{kj}\right) + \sum_{k,\beta}h^\alpha_{ik}h^\beta_{kj}V_\beta.$$

Covariant differentiating (5.9) over $M\times\{t\}$ and using the Codazzi equation (2.7) for $x_t\colon M\to R^{n+p}(c)$, we get

$$a_{i\alpha,j} = V_{\alpha,ij} + \sum_k\left(V_{k,j}h^\alpha_{ik} + V_kh^\alpha_{ikj}\right) = V_{\alpha,ij} + \sum_k\left(V_{k,j}h^\alpha_{ik} + V_kh^\alpha_{ijk}\right).$$

Combining the above two equations, we prove Lemma 5.2.

Let $x_0\colon M\to R^{n+p}(c)$ be an n-dimensional submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. Suppose $r\in\{0,1,\ldots,n\}$, the definition of S_r for each even integer r is given in (1.4) or (3.3), then we have the following lemma

Lemma 5.3 $S_{r,k} = \sum_{i,j,\alpha} T_{r-1ij}^{\ \alpha}h^\alpha_{ijk}$ *for a fixed index k.*

Proof

$$\begin{aligned} S_{r,k} &= \frac{1}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r}}\delta^{i_1\ldots i_r}_{j_1\ldots j_r}(\langle B_{i_1j_1},B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}},B_{i_rj_r}\rangle)_{,k}\\ &= \frac{1}{(r-1)!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ \alpha}}\delta^{i_1\ldots i_r}_{j_1\ldots j_r}\langle B_{i_1j_1},B_{i_2j_2}\rangle\cdots\langle B_{i_{r-3}j_{r-3}},B_{i_{r-2}j_{r-2}}\rangle h^{\ \alpha}_{i_{r-1}j_{r-1}}h^{\ \alpha}_{i_rj_rk}\\ &= \sum_{i_r,j_r,\alpha}T_{r-1i_rj_r}^{\ \alpha}h^{\ \alpha}_{i_rj_rk}\\ &= \sum_{i,j,\alpha}T_{r-1ij}^{\ \alpha}h^\alpha_{ijk} \end{aligned}$$

□

Proof of Theorem 5.1 From (1.4) and (5.15), we have

$$\begin{aligned}\frac{\partial S_r}{\partial t} &= \frac{1}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r}} \delta^{i_1\ldots i_r}_{j_1\ldots j_r}\frac{\partial}{\partial t}(\langle B_{i_1j_1},B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}},B_{i_rj_r}\rangle)\\
&= \frac{r}{2}\frac{1}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ \alpha}} \delta^{i_1\ldots i_r}_{j_1\ldots j_r}\langle B_{i_1j_1},B_{i_2j_2}\rangle\cdots\langle B_{i_{r-3}j_{r-3}},B_{i_{r-2}j_{r-2}}\rangle\left(\frac{\partial h_{i_{r-1}j_{r-1}}^{\alpha}}{\partial t}h^{\alpha}_{i_rj_r}+h^{\alpha}_{i_{r-1}j_{r-1}}\frac{\partial h^{\alpha}_{i_rj_r}}{\partial t}\right)\\
&= r\frac{1}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ \alpha}} \delta^{i_1\ldots i_r}_{j_1\ldots j_r}\langle B_{i_1j_1},B_{i_2j_2}\rangle\cdots\langle B_{i_{r-3}j_{r-3}},B_{i_{r-2}j_{r-2}}\rangle h^{\alpha}_{i_{r-1}j_{r-1}}\frac{\partial h^{\alpha}_{i_rj_r}}{\partial t}\\
&= \frac{1}{(r-1)!}\sum_{\substack{i_1\ldots i_{r-1}i\\ j_1\ldots j_{r-1}j\\ \alpha}} \delta^{i_1\ldots i_{r-1}i}_{j_1\ldots j_{r-1}j}\langle B_{i_1j_1},B_{i_2j_2}\rangle\cdots\langle B_{i_{r-3}j_{r-3}},B_{i_{r-2}j_{r-2}}\rangle h^{\alpha}_{i_{r-1}j_{r-1}}\frac{\partial h^{\alpha}_{ij}}{\partial t}\\
&= \sum_{i,j,\alpha} T_{r-1}{}^{\alpha}_{ij}\frac{\partial h^{\alpha}_{ij}}{\partial t}\\
&= \sum_{i,j,\alpha} T_{r-1}{}^{\alpha}_{ij}\left[V_{\alpha,ij}+\sum_k(a_{ik}h^{\alpha}_{kj}+a_{jk}h^{\alpha}_{ki}+h^{\alpha}_{ijk}V_k)+\sum_{k,\beta}h^{\alpha}_{ik}h^{\beta}_{kj}V_\beta\right.\\
&\quad\left.+c\cdot\delta_{ij}V_\alpha+\sum_\beta a_{\alpha\beta}h^{\beta}_{ij}\right]. \qquad (5.16)\end{aligned}$$

We are now in a position to compute the terms in the right-hand side of (5.16) one by one.

$$\begin{aligned}\sum_{i,j,k,\alpha} T_{r-1}{}^{\alpha}_{ij}(a_{ik}h^{\alpha}_{kj}+a_{jk}h^{\alpha}_{ki}) &= 2\sum_{i,j,k,\alpha} T_{r-1}{}^{\alpha}_{ij}h^{\alpha}_{kj}a_{ik}\\
&\overset{(1)}{=} 2\sum_{i,k}(S_r\delta^i_k - T_{rk}^{\;i})a_{ik}\\
&= 0 \qquad (5.17)\end{aligned}$$

where equality (1) from (3.10) and the last equality of (5.17) from fact $a_{ik}=-a_{ki}$ and $T_{rk}^{\;i}=T_{ri}^{\;k}$.

We have by use of (3.10), (3.4) and (3.9)

$$\begin{aligned}\sum_{\alpha,\beta}\sum_{i,j,k} T_{r-1}{}^{\alpha}_{ij}h^{\alpha}_{ik}h^{\beta}_{kj}V_\beta &= \sum_\beta\sum_{j,k}(S_r\delta^k_j - T_{rj}^{\;k})h^{\beta}_{kj}V_\beta\\
&= n\sum_\beta S_rH^\beta V_\beta - \sum_{i,j,\beta}T_{rj}^{\;i}h^{\beta}_{ij}V_\beta\\
&= S_r\langle \mathbf{S}_1, V\rangle - (r+1)\langle\mathbf{S}_{r+1}, V\rangle, \qquad (5.18)\end{aligned}$$

$$
\begin{aligned}
c \cdot \sum_{i,j,\alpha} T_{r-1\,ij}^{\ \alpha}\delta_{ij}V_\alpha &= c \cdot \sum_i T_{r-1\,ii}^{\ \alpha}V_\alpha \\
&= c \cdot \sum_{i,j,\alpha} \frac{n-r+1}{r-1} T_{r-2j}^{\ i} h_{ij}^{\alpha} V_\alpha \\
&= c \cdot (n-r+1)\langle \mathbf{S}_{r-1}, V\rangle. \qquad (5.19)
\end{aligned}
$$

We have by use of (3.2) and $a_{\alpha\beta} = -a_{\beta\alpha}$

$$
\begin{aligned}
&\sum_{\alpha,\beta}\sum_{i,j} T_{r-1ij}^{\ \alpha} h_{ij}^{\beta} a_{\alpha\beta} \\
&= \frac{1}{(r-1)!} \sum_{\substack{i_1\cdots i_{r-1}i \\ j_1\cdots j_{r-1}j \\ \alpha,\beta}} \delta_{j_1\ldots j_{r-1}j}^{i_1\ldots i_{r-1}i} \langle B_{i_1j_1}, B_{i_2j_2}\rangle \cdots \langle B_{i_{r-3}j_{r-3}}, B_{i_{r-2}j_{r-2}}\rangle h_{i_{r-1}j_{r-1}}^{\ \alpha} h_{ij}^{\beta} a_{\alpha\beta} \\
&= \frac{1}{(r-1)!} \sum_{\substack{i_1\cdots i_{r-1}i \\ j_1\cdots j_{r-1}j \\ \alpha,\beta}} \delta_{j_1\ldots j_{r-1}j}^{i_1\ldots i_{r-1}i} \langle B_{i_1j_1}, B_{i_2j_2}\rangle \cdots \langle B_{i_{r-3}j_{r-3}}, B_{i_{r-2}j_{r-2}}\rangle h_{ij}^{\alpha} h_{i_{r-1}j_{r-1}}^{\ \beta} a_{\alpha\beta} \\
&= 0. \qquad (5.20)
\end{aligned}
$$

Putting (5.17), (5.18), (5.19) and (5.20) into (5.16), we get (5.3) by use Lemma 5.3 and (4.2).

By use of (5.8), we have

Lemma 5.4

$$
\frac{\partial}{\partial t} dv_{g_t} = \frac{\partial}{\partial t}(\theta_1 \wedge \cdots \wedge \theta_n) = \sum_i V_{i,i} dv_{g_t} - \langle \mathbf{S}_1, V\rangle\, dv_{g_t}. \qquad (5.21)
$$

Combining (5.3) with (5.21), we can get (5.4) by use of (4.3). So we complete the proof of Proposition 5.1.

Proof of Theorem 1.1 We prove Theorem 1.1 by inductive method. From Lemma 5.4, we know that Theorem 1.1 is true for $r = 0$. Now we suppose that Theorem 1.1 is true for $r = k-2$ and $k-2$ is even, then we have

$$
{J_{k-2}}'(t) = -(k-1)\int_{M_t} \langle \mathbf{S}_{k-1}, V\rangle dv_{g_t}. \qquad (5.22)
$$

By use of (5.4) and (5.22), the following calculation shows us that Theorem 1.1 is true for $r = k =$ even

$$
\begin{aligned}
{J_k}'(t) &= \frac{d}{dt}\int_{M_t} F_k dv_{g_t} = \frac{d}{dt}\int_{M_t}\left(S_k + \frac{(n-k+1)c}{k-1}F_{k-2}\right) dv_{g_t} \\
&= \int_{M_t} \langle -(k+1)\mathbf{S}_{k+1} + (n-k+1)c\mathbf{S}_{k-1}, V\rangle dv_{g_t} \\
&\quad + \frac{(n-k+1)c}{k-1}\left[-(k-1)\int_M \langle \mathbf{S}_{k-1}, V\rangle dv_{g_t}\right] \\
&= -(k+1)\int_{M_t} \langle \mathbf{S}_{k+1}, V\rangle dv_{g_t}. \qquad (5.23)
\end{aligned}
$$

Thus we complete the proof of Theorem 1.1.

From Theorem 1.1, we know that $J_r'(0) = 0$ if and only if $\mathbf{S}_{r+1} \equiv 0$ on M, which leads us to the following definition.

Definition 5.1 Let M be an n-dimensional submanifold in $R^{n+p}(c)$. Assume that r is even and $r \in \{0, 1, \ldots, n-1\}$, we call M to be an r-minimal submanifold if the $(r+1)$th mean curvature vector $\mathbf{S}_{r+1} \equiv 0$ on M.

Remark 5.3 When $p = 1$, let $x_0 \colon M \to R^{n+1}(c)$ be an n-dimensional compact hypersurface in $R^{n+1}(c)$, Barbosa–Colares [5] (also see [1, 3]) considered the variational problem keeping the balance of volume for any integer r.

Now we would like to present a family of examples of r-minimal submanifolds in space form. For convenience we assume that r is any even integer and $r \in \{0, 1, \ldots, n-1\}$ in the following examples.

Example 5.1 Totally geodesic submanifolds in space form are r-minimal.

Example 5.2 Let C^{n+p} be a (complex) $(n+p)$-dimensional Euclidean space, assume $x \colon M \to C^{n+p}$ is a Kähler submanifold of (complex) dimension n. Then M is r-minimal.

Proof Let $e_A = \{e_1, \ldots, e_{n+p}; e_{1^*} = Je_1, \ldots, e^*_{n+p} = Je_{n+p}\}$ be a local orthonormal basis for C^{n+p}. We make use the notation of $e_{(A^*)^*} = -e_A, \delta_{A^*C} = -\delta_{AC^*}$ and the indices range convention $A, C, A^*, C^* = 1, \ldots, n+p, 1^*, \ldots, (n+p)^*$.

We can choose e_A such that, restricted to M, $\{e_1, \ldots, e_n; e_{1^*} = Je_1, \ldots, e_{n^*} = Je_n\}$ is a local orthonormal basis for TM. We further use the following convention on the range of indices:

$$\alpha, \beta, \alpha^*, \beta^* = n+1, \ldots, n+p, (n+1)^*, \ldots, (n+p)^*.$$

Then the second fundamental form $B_{A^*C^*}$ and B_{AC} of the immersion x have the following relations (see [15])

$$\begin{aligned} B_{A^*C^*} = B(e_{A^*}, e_{C^*}) &= \sum_\alpha (h^\alpha_{A^*C^*} e_\alpha + h^{\alpha^*}_{A^*C^*} e_{\alpha^*}) \\ &= \sum_\alpha (-h^\alpha_{AC} e_\alpha - h^{\alpha^*}_{AC} e_{\alpha^*}) = -B_{AC}. \end{aligned}$$

Since $\delta_{A^*C^*} = \langle e_{A^*}, e_{C^*}\rangle = \langle Je_A, Je_C\rangle = \langle e_A, e_C\rangle = \delta_{AC}$, by the definition of the generalized Kronecker symbols $\delta^{A_1\ldots A_{r+1}}_{C_1\ldots C_{r+1}}$ in Sect. 3, we have $\delta^{A_1\ldots A_{r+1}}_{C_1\ldots C_{r+1}} = \delta^{A_1^*\ldots A_{r+1}^*}_{C_1^*\ldots C_{r+1}^*}$.

For any fixed index $A_1, \ldots, A_r, A$, we have

$$\begin{aligned} &\delta^{A_1^*\ldots A_r^* A^*}_{C_1^*\ldots C_r^* C^*} \langle B_{A_1^*C_1^*}, B_{A_2^*C_2^*}\rangle \cdots \langle B_{A_{r-1}^*C_{r-1}^*}, B_{A_r^*C_r^*}\rangle B_{A^*C^*} \\ &\quad = \delta^{A_1\ldots A_r A}_{C_1\ldots C_r C} \langle -B_{A_1C_1}, -B_{A_2C_2}\rangle \cdots \langle -B_{A_{r-1}C_{r-1}}, -B_{A_rC_r}\rangle (-B_{AC}) \\ &\quad = -\delta^{A_1\ldots A_r A}_{C_1\ldots C_r C} \langle B_{A_1C_1}, B_{A_2C_2}\rangle \cdots \langle B_{A_{r-1}C_{r-1}}, B_{A_rC_r}\rangle B_{AC} \end{aligned}$$

Thus

$$
\begin{aligned}
\mathbf{S}_{r+1} &= \frac{1}{(r+1)!}\sum_{\substack{A_1\ldots A_rA\\ C_1\ldots C_rC}} \delta^{A_1\ldots A_rA}_{C_1\ldots C_rC}\langle B_{A_1C_1},B_{A_2C_2}\rangle\cdots\langle B_{A_{r-1}C_{r-1}},B_{A_rC_r}\rangle B_{AC}\\
&= \frac{1}{(r+1)!}\sum_{\substack{A^*_1\ldots A^*_rA^*\\ C^*_1\ldots C^*_rC*}} \delta^{A^*_1\ldots A^*_rA^*}_{C^*_1\ldots C^*_rC^*}\langle B_{A^*_1C^*_1},B_{A^*_2C^*_2}\rangle\cdots\langle B_{A^*_{r-1}C^*_{r-1}},B_{A^*_rC^*_r}\rangle B_{A^*C^*}\\
&= -\frac{1}{(r+1)!}\sum_{\substack{A_1\ldots A_rA\\ C_1\ldots C_rC}} \delta^{A_1\ldots A_rA}_{C_1\ldots C_rC}\langle B_{A_1C_1},B_{A_2C_2}\rangle\cdots\langle B_{A_{r-1}C_{r-1}},B_{A_rC_r}\rangle B_{AC}\\
&= 0
\end{aligned}
$$

Thus $x\colon M^n \to C^{n+p}$ is $r-$minimal. □

Example 5.3 Let $M^n(k)$ and $N^{n+p}(c)$ be n-dimensional and $(n+p)$-dimensional space forms with constant sectional curvature k and c, respectively. If the immersion $x\colon M^n(k)\to N^{n+p}(c)$ is minimal, then x is r-minimal.

Proof By the minimal condition, we have $\sum_i B_{ii}=0$.

In the present case, from (2.4) we get the Gauss equation

$$\langle B_{ik},B_{jl}\rangle-\langle B_{il},B_{jk}\rangle=(k-c)(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})$$

Thus we have

$$
\begin{aligned}
\mathbf{S}_{r+1} &= \frac{1}{(r+1)!}\sum_{\substack{i_1\ldots i_ri\\ j_1\ldots j_rj}} \delta^{i_1\ldots i_ri}_{j_1\ldots j_rj}\langle B_{i_1j_1},B_{i_2j_2}\rangle\cdots\langle B_{i_{r-1}j_{r-1}},B_{i_rj_r}\rangle B_{ij}\\
&= \frac{1}{(r+1)!}\cdot\frac{1}{2^{\frac{r}{2}}}\sum_{\substack{i_1\ldots i_ri\\ j_1\ldots j_rj}} \delta^{i_1\ldots i_ri}_{j_1\ldots j_rj}(\langle B_{i_1j_1},B_{i_2j_2}\rangle-\langle B_{i_1j_2},B_{i_2j_1}\rangle)\cdots(\langle B_{i_{r-1}j_{r-1}},B_{i_rj_r}\rangle\\
&\quad-\langle B_{i_{r-1}j_r},B_{i_rj_{r-1}}\rangle)B_{ij}\\
&= \frac{1}{(r+1)!}\cdot\frac{1}{2^{\frac{r}{2}}}\cdot(k-c)^{\frac{r}{2}}\sum_{\substack{i_1\ldots i_ri\\ j_1\ldots j_rj}} \delta^{i_1\ldots i_ri}_{j_1\ldots j_rj}(\delta_{i_1j_1}\delta_{i_2j_2}-\delta_{i_1j_2}\delta_{i_2j_1})\cdots(\delta_{i_{r-1}j_{r-1}}\delta_{i_rj_r}\\
&\quad-\delta_{i_{r-1}j_r}\delta_{i_rj_{r-1}})B_{ij}\\
&= \frac{(k-c)^{\frac{r}{2}}}{(r+1)!}\sum_{\substack{i_1\ldots i_ri\\ j_1\ldots j_rj}} \delta^{i_1\ldots i_ri}_{j_1\ldots j_rj}\delta_{i_1j_1}\delta_{i_2j_2}\cdots\delta_{i_{r-1}j_{r-1}}\delta_{i_rj_r}B_{ij}\\
&= \frac{(k-c)^{\frac{r}{2}}}{(r+1)!}(n-r)(n-r+1)\cdots(n-1)\sum_{ij}\delta_{ij}B_{ij}\\
&= \frac{(k-c)^{\frac{r}{2}}}{(r+1)!}(n-r)(n-r+1)\cdots(n-1)\sum_{i}B_{ii}\\
&= 0
\end{aligned}
$$

□

Remark 5.4 In particular, the submanifold $x : S^n\left(\sqrt{\frac{2(n+1)}{n}}\right) \to S^{n+p}(1)$ with $p = \frac{1}{2}(n-1)(n+2)$ is r-minimal. (see [13])

Example 5.4 $M_r = S^1\left(\sqrt{\frac{r+1}{n}}\right) \times S^{n-1}\left(\sqrt{\frac{n-r-1}{n}}\right) \to S^{n+1}(1)$ is r-minimal. Among these r-minimal hypersurfaces, the only minimal hypersurface is the case with $r = 0$.

Proof Suppose $M_r = S^1(a) \times S^{n-1}(b)$ is an n-dimensional compact r-minimal hypersurface in S^{n+1}, then $a^2 + b^2 = 1$.

Assume that two distinct principal curvatures of M_r are λ (multiplicity 1) and μ (multiplicity $n-1$). Then (see [13]) we have

$$1 + \lambda\mu = 0. \tag{5.24}$$

By the condition of r-minimal, we have $\binom{n-1}{r}\lambda\mu^{r-1} + \binom{n-1}{r+1}\mu^r = 0$, i.e.

$$(1+r)\lambda + (n-r-1)\mu = 0 \tag{5.25}$$

We have by combining (5.24) and (5.25)

$$\lambda = \sqrt{\frac{n-r-1}{r+1}}, \quad \mu = -\sqrt{\frac{r+1}{n-r-1}}.$$

Then

$$a^2 = \frac{1}{1+\lambda^2} = \frac{r+1}{n}, \quad b^2 = \frac{1}{1+\mu^2} = \frac{n-r-1}{n},$$

$$M_r = S^1\left(\sqrt{\frac{r+1}{n}}\right) \times S^{n-1}\left(\sqrt{\frac{n-r-1}{n}}\right).$$

If M_r is minimal, we have

$$nH = \sqrt{\frac{n-r-1}{r+1}} - (n-1)\sqrt{\frac{r+1}{n-r-1}} = 0.$$

We obtain $r = 0$. □

Example 5.5 $C_{n,n} = S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}}) \to S^{2n+1}(1)$ is r-minimal.

We all know that $C_{n,n}$ is minimal in S^{2n+1} and its principal curvatures $k_1, \ldots, k_{2n}$ are

$$k_1 = \cdots = k_n = 1, \quad k_{n+1} = \cdots = k_{2n} = -1.$$

Thus

$$\begin{aligned} S_{r+1} &= \sum_{i_1<i_2<\cdots<i_{r+1}} k_{i_1}k_{i_2}\ldots k_{i_{r+1}} \\ &= \sum_{k=0}^{n}\binom{n}{k}\binom{n}{r+1-k}1^k(-1)^{(r+1)-k} + \sum_{k=0}^{n}\binom{n}{k}\binom{n}{r+1-k}(-1)^k 1^{(r+1)-k} \\ &= \sum_{k=0}^{n}\binom{n}{k}\binom{n}{r+1-k}((-1)^{(r+1)-k}+(-1)^k) \\ &= 0 \end{aligned}$$

The last equality is by the fact that $(-1)^{(r+1)-k}+(-1)^k=0$ when the integer r is even, $k=0,\ldots,r+1$.

Example 5.6 Let $x\colon M\to N^{n+p}(c)$ be an n-dimensional minimal immersion in $(n+p)$-dimensional space form $N^{n+p}(c)$. If M is an Einstein manifold, then $x\colon M\to N^{n+p}(c)$ is 2-minimal.

Proof We have by the assumption

$$\sum_i B_{ii}=0, \quad R_{ij}=\frac{R}{n}\delta_{ij} \tag{5.26}$$

where R is the scalar curvature of M^n.

Put the condition (5.26) into the Gauss equation (2.5), we have

$$\sum_k \langle B_{ik},B_{kj}\rangle = \left((n-1)c-\frac{R}{n}\right)\delta_{ij}$$

Then

$$\begin{aligned} \mathbf{S_3} &= \frac{1}{3!}\sum_{\substack{i_1i_2i\\ j_1j_2j}} \delta^{i_1i_2i}_{j_1j_2j}\langle B_{i_1j_1},B_{i_2j_2}\rangle B_{ij} \\ &= \frac{1}{3}\sum_{ij}\left[\delta^i_j S_2 - \sum_{\substack{i_1i_2\\ j_1j_2}} \delta^{\,i}_{j_2}\,\delta^{i_1i_2}_{j_1j}\langle B_{i_1j_1},B_{i_2j_2}\rangle\right]B_{ij} \\ &= \frac{1}{3}\left[S_2\cdot\sum_i B_{ii} - \sum_{\substack{i_1i_2i\\ j_1j_2j}}(\delta^{i_1}_{j_1}\delta^{i_2}_{j} - \delta^{i_1}_{j}\delta^{i_2}_{j_1})\langle B_{i_1j_1},B_{i_2i}\rangle B_{ij}\right] \\ &= \frac{1}{3}\left[-\sum_{i,j}\left(\sum_k\langle B_{kk},B_{ij}\rangle - \sum_k\langle B_{jk},B_{ki}\rangle\right)B_{ij}\right] \\ &= \frac{1}{3}\sum_{i,j,k}\langle B_{jk},B_{ki}\rangle B_{ij} \\ &= \frac{1}{3}\left((n-1)c-\frac{R}{n}\right)\cdot\sum_i B_{ii} \\ &= 0 \end{aligned}$$

Therefore $x\colon M^n \to N^{n+p}(c)$ is 2-minimal. □

Example 5.7 Let $k_1 > k_2 > \cdots > k_g$ be the distinct principal curvatures with multiplicities $m_1, \ldots, m_g$ (so that $n = m_1 + m_2 + \cdots + m_g$) of n-dimensional isoparametric hypersurfaces M in S^{n+1}. Suppose M is one of the following cases, then M is r-minimal in S^{n+1}.

(1) If $g = 3$ and $k_1 = \sqrt{3},\ k_2 = 0,\ k_3 = -\sqrt{3};\ m_1 = m_2 = m_3 = 2^k (k = 0, 1, 2, 3)$.
(2) If $g = 4$ and $k_1 = 1 + \sqrt{2},\ k_2 = \sqrt{2} - 1,\ k_3 = 1 - \sqrt{2},\ k_4 = -(1 + \sqrt{2})$ with $m_1 = m_2 = m_3 = m_4 = 2$. Now the hypersurfaces are Cartan minimal hypersurface.
(3) If $g = 6$ and $k_1 = 2 + \sqrt{3},\ \ k_2 = 1,\ \ k_3 = 2 - \sqrt{3},\ \ k_4 = -(2 - \sqrt{3}),\ \ k_5 = -1,$ $k_6 = -(2 + \sqrt{3});\ m_1 = \cdots = m_6 = 1$ or 2.

Firstly, we give an algebraic result. Let M^n be an n-dimensional compact hypersurface of the $(n+1)$-dimensional Riemannian manifold N^{n+1}. The principal curvatures of the hypersurface M is denoted by $k_1, \ldots, k_n$. Write

$$S_0 = 1, \quad S_r = \sum_{i_1 < i_2 < \cdots < i_r} k_{i_1} k_{i_2} \ldots k_{i_r},$$

$$P_0 = n, \qquad P_r = \sum_i (k_i)^r \quad (r = 1, 2, \ldots, n).$$

We have $S_1, \ldots, S_n$ denoted by use of $P_1, \ldots P_n$

$$S_1 = P_1,$$

$$S_2 = \frac{1}{2!}(-P_2 + P_1^2),$$

$$S_3 = \frac{1}{3!}(2P_3 - 3P_1P_2 + P_1^3),$$

$$\ldots$$

$$S_l = \sum_{\substack{t_1 + 2t_2 + \cdots + lt_l = l \\ 0 \le t_i}} \frac{(-1)^{t_1 + t_2 + \cdots + t_l + l}}{(t_1!) \ldots (t_l!) 2^{t_2} \ldots l^{t_l}} P_1^{t_1} P_2^{t_2} \ldots P_l^{t_l},$$

$$\ldots$$

$$S_n = \sum_{\substack{t_1 + 2t_2 + \cdots + nt_n = n \\ 0 \le t_i}} \frac{(-1)^{t_1 + t_2 + \ldots + t_n + n}}{(t_1!) \ldots (t_l!) 2^{t_2} \ldots n^{t_n}} P_1^{t_1} P_2^{t_2} \ldots P_n^{t_n}.$$

Now we consider hypersurfaces (1)–(3) in S^{n+1} above, it is easy to verify that

$$P_1 = P_3 = \cdots = P_{r+1} = 0 \tag{5.27}$$

for the even integer r.

On the other side, when $l = r+1$,

$$S_{r+1} = \sum_{\substack{t_1+2t_2+\cdots+(r+1)t_{r+1}=r+1\\ 0\le t_i}} \frac{(-1)^{t_1+t_2+\cdots+t_{r+1}+(r+1)}}{(t_1!)\ldots(t_{r+1}!)2^{t_2}\ldots(r+1)^{t_{r+1}}} P_1^{t_1}P_2^{t_2}\ldots P_{r+1}^{t_{r+1}}. \quad (5.28)$$

We observe that $t_1 + 2t_2 + \cdots + (r+1)t_{r+1} = \sum_{i=0}^{r+1} i \cdot t_i = r+1$ is odd, then there exists a factor $i \cdot t_i$ such that $i \cdot t_i$ is an odd integer, that is, both i and t_i are odd, we have by (5.27)

$$P_i^{t_i} = 0 \quad (5.29)$$

where $i \in \{0, 1, \ldots, r+1\}$.

Since each term $P_1^{t_1}P_2^{t_2}\cdots P_{r+1}^{t_{r+1}}$ of S_{r+1} includes such factor as (5.29), we have $S_{r+1} = 0$, thus we prove that (1)–(3) in example 5.7 are r-minimal.

Remark 5.5 In [19], (1)–(3) in Example 5.7 are given as minimal Willmore hypersurfaces in S^{n+1}.

6 The second variational formula

In this section, suppose $x_0 : M \to R^{n+p}(c)$ be an n-dimensional compact r-minimal submanifold in $R^{n+p}(c)$ for some even integer $r \in \{0, 1, \ldots, n-1\}$, we will calculate the second variational formula of the functional $J_r(t)$.

Before stating the second variational formula, we still need to introduce some notations when r is even. Define

$$T_{r j_{r+1}\, j_{r+2}}^{\;\, i_{r+1}\, i_{r+2}} = \frac{1}{r!} \sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r}} \delta^{i_1\ldots i_r i_{r+1} i_{r+2}}_{j_1\ldots j_r j_{r+1} j_{r+2}} \langle B_{i_1j_1}, B_{i_2j_2}\rangle \cdots \langle B_{i_{r-1}j_{r-1}}, B_{i_rj_r}\rangle \quad (6.1)$$

$$T_{r\, j_{r+1}j_{r+2}\, j}^{\;\, i_{r+1}i_{r+2}\, i} = \frac{1}{r!} \sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r}} \delta^{i_1\ldots i_r i_{r+1} i_{r+2} i}_{j_1\ldots j_r j_{r+1} j_{r+2} j} \langle B_{i_1j_1}, B_{i_2j_2}\rangle \cdots \langle B_{i_{r-1}j_{r-1}}, B_{i_rj_r}\rangle \quad (6.2)$$

Lemma 6.1

$$\begin{aligned} T_{r\, j_{r+1}j_{r+2}}^{\;\, i_{r+1}i_{r+2}} &= \delta^{i_{r+2}}_{j_{r+2}} T_{r j_{r+1}}^{\;\, i_{r+1}} - \delta^{i_{r+2}}_{j_{r+1}} T_{r j_{r+2}}^{\;\, i_{r+1}} \\ &\quad - \frac{1}{r-1} \sum_{\alpha} \sum_{\substack{i_{r-1} i_r\\ j_{r-1}}} T_{r-2\, j_{r-1}j_{r+1}j_{r+2}}^{\;\;\, i_{r-1}i_{r+1}\, i_r} h^{\alpha}_{i_{r-1}j_{r-1}} h^{\alpha}_{i_r i_{r+2}}. \end{aligned} \quad (6.3)$$

Proof By Definition (6.1) and the property of $\delta^{i_1\ldots i_r i_{r+1} i_{r+2}}_{j_1\ldots j_r j_{r+1} j_{r+2}}$, we have

$$
\begin{aligned}
T_r{}^{i_{r+1}i_{r+2}}_{j_{r+1}j_{r+2}} &= \frac{1}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r}} \delta^{i_1\ldots i_r i_{r+1} i_{r+2}}_{j_1\ldots j_r j_{r+1} j_{r+2}} \langle B_{i_1j_1}, B_{i_2j_2}\rangle \cdots \langle B_{i_{r-1}j_{r-1}}, B_{i_rj_r}\rangle \\
&= \delta^{i_{r+2}}_{j_{r+2}} T_r{}^{i_{r+1}}_{j_{r+1}} - \delta^{i_{r+2}}_{j_{r+1}} T_r{}^{i_{r+1}}_{j_{r+2}} + \frac{1}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ \alpha}} \delta^{i_{r+2}}_{j_r}\delta^{i_1\ldots i_{r-1}\ i_r\ i_{r+1}}_{j_1\ldots j_{r-1} j_{r+1} j_{r+2}} \langle B_{i_1j_1}, B_{i_2j_2}\rangle \\
&\quad \cdots \langle B_{i_{r-3}j_{r-3}}, B_{i_{r-2}j_{r-2}}\rangle h^{\alpha}_{i_{r-1}j_{r-1}} h^{\alpha}_{i_rj_r} - \frac{1}{r!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ \alpha}} \delta^{i_{r+2}}_{j_{r-1}}\delta^{i_1\ldots i_{r-2} i_{r-1}\ i_r\ i_{r+1}}_{j_1\ldots j_{r-2}\ j_r\ j_{r+1} j_{r+2}} \\
&\quad \cdot\langle B_{i_1j_1}, B_{i_2j_2}\rangle \cdots \langle B_{i_{r-3}j_{r-3}}, B_{i_{r-2}j_{r-2}}\rangle h^{\alpha}_{i_{r-1}j_{r-1}} h^{\alpha}_{i_rj_r} + \cdots \\
&= \delta^{i_{r+2}}_{j_{r+2}} T_r{}^{i_{r+1}}_{j_{r+1}} - \delta^{i_{r+2}}_{j_{r+1}} T_r{}^{i_{r+1}}_{j_{r+2}} + \frac{1}{r(r-1)}\sum_{\alpha}\sum_{\substack{i_{r-1}i_r\\ j_{r-1}}} T_{r-2}{}^{i_{r-1}\ i_r\ i_{r+1}}_{j_{r-1}j_{r+1}j_{r+2}} h^{\alpha}_{i_{r-1}j_{r-1}} h^{\alpha}_{i_ri_{r+2}} \\
&\quad -\frac{1}{r(r-1)}\sum_{\alpha}\sum_{\substack{i_{r-1}i_r\\ j_{r-1}}} T_{r-2}{}^{i_{r-1}\ i_r\ i_{r+1}}_{\ j_r\ j_{r+1}j_{r+2}} h^{\alpha}_{i_{r-1}i_{r+2}} h^{\alpha}_{i_rj_r} + \cdots \\
&= \delta^{i_{r+2}}_{j_{r+2}} T_r{}^{i_{r+1}}_{j_{r+1}} - \delta^{i_{r+2}}_{j_{r+1}} T_r{}^{i_{r+1}}_{j_{r+2}} - \frac{1}{r(r-1)}\sum_{\alpha}\sum_{\substack{i_{r-1}i_r\\ j_{r-1}}} T_{r-2}{}^{i_{r-1}i_{r+1}\ i_r}_{j_{r-1}j_{r+1}j_{r+2}} h^{\alpha}_{i_{r-1}j_{r-1}} h^{\alpha}_{i_ri_{r+2}} \\
&\quad - \frac{1}{r(r-1)}\sum_{\alpha}\sum_{\substack{i_{r-1}i_r\\ j_{r-1}}} T_{r-2}{}^{i_r\ i_{r-1}\ i_{r+1}}_{j_{r-1}j_{r+1}j_{r+2}} h^{\alpha}_{i_ri_{r+2}} h^{\alpha}_{i_{r-1}j_{r-1}} + \cdots \\
&= \delta^{i_{r+2}}_{j_{r+2}} T_r{}^{i_{r+1}}_{j_{r+1}} - \delta^{i_{r+2}}_{j_{r+1}} T_r{}^{i_{r+1}}_{j_{r+2}} - \frac{1}{r(r-1)}\sum_{\alpha}\sum_{\substack{i_{r-1}i_r\\ j_{r-1}}} T_{r-2}{}^{i_{r-1}i_{r+1}i_r}_{j_{r-1}j_{r+1}j_{r+2}} h^{\alpha}_{i_{r-1}j_{r-1}} h^{\alpha}_{i_ri_{r+2}} \\
&\quad - \frac{1}{r(r-1)}\sum_{\alpha}\sum_{\substack{i_{r-1}i_r\\ j_{r-1}}} T_{r-2}{}^{i_{r-1}i_{r+1}i_r}_{j_{r-1}j_{r+1}j_{r+2}} h^{\alpha}_{i_{r-1}j_{r-1}} h^{\alpha}_{i_ri_{r+2}} + \cdots \\
&= \delta^{i_{r+2}}_{j_{r+2}} T_r{}^{i_{r+1}}_{j_{r+1}} - \delta^{i_{r+2}}_{j_{r+1}} T_r{}^{i_{r+1}}_{j_{r+2}} - \frac{r}{r(r-1)}\sum_{\alpha}\sum_{\substack{i_{r-1}i_r\\ j_{r-1}}} T_{r-2}{}^{i_{r-1}i_{r+1}\ i_r}_{j_{r-1}j_{r+1}j_{r+2}} h^{\alpha}_{i_{r-1}j_{r-1}} h^{\alpha}_{i_ri_{r+2}} \\
&= \delta^{i_{r+2}}_{j_{r+2}} T_r{}^{i_{r+1}}_{j_{r+1}} - \delta^{i_{r+2}}_{j_{r+1}} T_r{}^{i_{r+1}}_{j_{r+2}} - \frac{1}{(r-1)}\sum_{\alpha}\sum_{\substack{i_{r-1}i_r\\ j_{r-1}}} T_{r-2}{}^{i_{r-1}i_{r+1}\ i_r}_{j_{r-1}j_{r+1}j_{r+2}} h^{\alpha}_{i_{r-1}j_{r-1}} h^{\alpha}_{i_ri_{r+2}}. \quad \square
\end{aligned}
$$

Theorem 6.1 (the second variational formula) *Let M be an n-dimensional compact r-minimal submanifold in $R^{n+p}(c)$ for some even integer $r \in \{0, 1, \ldots, n-1\}$, we have*

$$
J''(0) = -\int_M \sum_{\alpha} V_{\alpha} \left\{ \frac{1}{r-1} \sum_{\substack{i_{r-1}i_ri\\ j_{r-1}j_rj\\ \beta}} T_{r-2}{}^{i_{r-1}i_ri}_{j_{r-1}j_rj} h^{\alpha}_{i_{r-1}j_{r-1}} h^{\beta}_{i_rj_r} V_{\beta,ij} \right.
$$

$$+\sum_{i,j} T_{rj}^{\ i} V_{\alpha,ij} + c \cdot (n-r)(S_r V_\alpha + \sum_{i,j,\beta} T_{r-1\ ij}^{\ \alpha} h_{ij}^{\beta} V_\beta)$$

$$\left. -(r+1) \sum_{i,j,\beta} T_{r+1\ ij}^{\ \alpha} h_{ij}^{\beta} V_\beta \right\} dv \tag{6.4}$$

When $p = 1$, we get by use of (3.9) and (3.10)

Corollary 6.1 (c.f. [5]) *Assume that r is even and $r \in \{0, 1, \ldots, n-1\}$. Let M be an n-dimensional compact r-minimal hypersurface in $R^{n+1}(c)$, then we have*

$$J_r''(0) = -(r+1) \int_M \lambda [L_r(\lambda) + c \cdot (n-r) S_r \lambda - (r+2) S_{r+2} \lambda] dv, \tag{6.5}$$

where $V = \lambda e_{n+1}$ and L_r is defined in section 4.

Proof of Theorem 6.1 Because M is r-minimal, thus it satisfies

$$\sum_{i,j} T_{rj}^{\ i} h_{ij}^{\alpha} = 0, \quad n+1 \le \alpha \le n+p. \tag{6.6}$$

By use of Theorem 1.1, we have

$$J_r''(0) = -(r+1) \int_M \left\langle \frac{\partial \mathbf{S}_{r+1}}{\partial t} |_{t=0}, V \right\rangle dv. \tag{6.7}$$

Putting (3.4) into (6.7), we have by use of (6.6)

$$\begin{aligned} J_r''(0) &= -(r+1) \int_M \left\langle \frac{\partial \mathbf{S}_{r+1}}{\partial t} |_{t=0}, V \right\rangle dV \\ &= -\int_M \left\langle \sum_{i,j,\alpha} \frac{\partial (T_{rj}^{\ i} h_{ij}^{\alpha} e_\alpha)}{\partial t} |_{t=0}, V \right\rangle dV \\ &= -\int_M \left\{ \sum_{i,j,\alpha} \frac{\partial (T_{rj}^{\ i} h_{ij}^{\alpha})}{\partial t} |_{t=0} V_\alpha \right\} dv. \end{aligned} \tag{6.8}$$

By use of (3.1) and (5.15), we have the following calculation at $t = 0$

$$
\begin{aligned}
\sum_{i,j,\alpha} \frac{\partial (T_{r\,j}^{\ i} h_{ij}^{\alpha})}{\partial t} V_\alpha &= \sum_{i,j,\alpha} \left(\frac{\partial T_{r\,j}^{\ i}}{\partial t} h_{ij}^{\alpha} + T_{r\,j}^{\ i} \frac{\partial h_{ij}^{\alpha}}{\partial t} \right) V_\alpha \\
&= \sum_{i,j,\alpha} \frac{\partial}{\partial t} \left(\frac{1}{r!} \sum_{\substack{i_1 \ldots i_r \\ j_1 \ldots j_r}} \delta_{j_1 \ldots j_r j}^{i_1 \ldots i_r i} \langle B_{i_1 j_1}, B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}}, B_{i_r j_r} \rangle \right) h_{ij}^{\alpha} V_\alpha \\
&\quad + \sum_{i,j,\alpha} T_{r\,j}^{\ i} \frac{\partial h_{ij}^{\alpha}}{\partial t} V_\alpha \\
&= \frac{1}{r-1} \sum_{\substack{i_{r-1} i_r i \\ j_{r-1} j_r j \\ \alpha,\beta}} T_{r-2}{}_{\,j_{r-1} j_r j}^{\,i_{r-1} i_r i} h_{i_{r-1} j_{r-1}}^{\ \beta} \frac{h_{i_r j_r}^{\ \beta}}{\partial t} h_{ij}^{\alpha} V_\alpha + \sum_{i,j,\alpha} T_{r\,j}^{\ i} \frac{\partial h_{ij}^{\alpha}}{\partial t} V_\alpha \\
&= \frac{1}{r-1} \sum_{\substack{i_{r-1} i_r i \\ j_{r-1} j_r j \\ \alpha,\beta}} T_{r-2}{}_{\,j_{r-1} j_r j}^{\,i_{r-1} i_r i} h_{i_{r-1} j_{r-1}}^{\ \alpha} \frac{h_{i_r j_r}^{\ \alpha}}{\partial t} h_{ij}^{\beta} V_\beta + \sum_{i,j,\alpha} T_{r\,j}^{\ i} \frac{\partial h_{ij}^{\alpha}}{\partial t} V_\alpha \\
&= \frac{1}{r-1} \sum_{\substack{i_{r-1} i_r i \\ j_{r-1} j_r j \\ \alpha,\beta}} T_{r-2}{}_{\,j_{r-1} j_r j}^{\,i_{r-1} i_r i} h_{i_{r-1} j_{r-1}}^{\ \alpha} h_{i_r j_r}^{\ \beta} \frac{h_{ij}^{\alpha}}{\partial t} V_\beta + \sum_{i,j,\alpha} T_{r\,j}^{\ i} \frac{\partial h_{ij}^{\alpha}}{\partial t} V_\alpha \\
&= \sum_{i,j,\alpha} \left(\frac{1}{r-1} \sum_{\substack{i_{r-1} i_r \\ j_{r-1} j_r \\ \beta}} T_{r-2}{}_{\,j_{r-1} j_r j}^{\,i_{r-1} i_r i} h_{i_{r-1} j_{r-1}}^{\ \alpha} h_{i_r j_r}^{\ \beta} V_\beta + T_{r\,j}^{\ i} V_\alpha \right) \frac{\partial h_{ij}^{\alpha}}{\partial t} \\
&= \sum_{i,j,\alpha} \left(\frac{1}{r-1} \sum_{\substack{i_{r-1} i_r \\ j_{r-1} j_r \\ \beta}} T_{r-2}{}_{\,j_{r-1} j_r j}^{\,i_{r-1} i_r i} h_{i_{r-1} j_{r-1}}^{\ \alpha} h_{i_r j_r}^{\ \beta} V_\beta + T_{r\,j}^{\ i} V_\alpha \right) \Bigg(V_{\alpha,ij} + c \cdot \delta_{ij} V_\alpha \\
&\quad + \sum_k (a_{ik} h_{kj}^{\alpha} + a_{jk} h_{ki}^{\alpha} + h_{ijk}^{\alpha} V_k) + \sum_{k,\gamma} h_{ik}^{\alpha} h_{kj}^{\gamma} V_\gamma + \sum_\gamma a_{\alpha\gamma} h_{ij}^{\gamma} \Bigg). \qquad (6.9)
\end{aligned}
$$

Now let us compute the terms in the right hand side of (6.9) one by one.

$$
\sum_{i,j,\alpha} \left(\frac{1}{r-1} \sum_{\substack{i_{r-1} i_r \\ j_{r-1} j_r \\ \beta}} T_{r-2}{}_{\,j_{r-1} j_r j}^{\,i_{r-1} i_r i} h_{i_{r-1} j_{r-1}}^{\ \alpha} h_{i_r j_r}^{\ \beta} V_\beta + T_{r\,j}^{\ i} V_\alpha \right) \left(\sum_k (a_{ik} h_{kj}^{\alpha} + a_{jk} h_{ki}^{\alpha}) \right)
$$

$$
\begin{aligned}
&= 2\sum_{j,k} a_{jk}\left(\frac{1}{r-1}\sum_{\substack{i_{r-1}i_r\\ j_{r-1}j_r\\ \alpha,\beta}} T_{r-2}{}^{i_{r-1}i_r i}_{j_{r-1}j_r j} h^{\ \alpha}_{i_{r-1}j_{r-1}} h^{\alpha}_{ik} h^{\ \beta}_{i_r j_r} V_\beta + \sum_{i,\alpha} T_r{}^{i}_{j} h^{\alpha}_{ik} V_\alpha\right)\\
&= 2\sum_{j,k} a_{jk}\sum_{\substack{i_r j_r\\ \beta}}\left(-T_r{}^{i_r k}_{j_r j} + \delta^k_j T_r{}^{i_r}_{j_r} - \delta^k_{j_r} T_r{}^{i_r}_{j}\right) h^{\ \beta}_{i_r j_r} V_\beta + 2\sum_{\alpha}\sum_{i,j,k} a_{jk} T_r{}^{i}_{j} h^{\alpha}_{ik} V_\alpha\\
&= 2\sum_{j,k} a_{jk}\left(-(r+1)\sum_{\alpha} T_{r+1}{}^{\alpha}_{kj} V_\alpha + \delta^k_j \sum_{i_r, j_r, \beta} T_r{}^{i_r}_{j_r} h^{\ \beta}_{i_r j_r} V_\beta\right)\\
&\overset{(1)}{=} 0.
\end{aligned}
\tag{6.10}
$$

where equality (1) from (3.7) and $a_{jk} = -a_{kj}$.

We have by use of (3.1) and (6.6)

$$
\begin{aligned}
&\sum_{i,j,k,\alpha}\left(\frac{1}{r-1}\sum_{\substack{i_{r-1}i_r\\ j_{r-1}j_r\\ \beta}} T_{r-2}{}^{i_{r-1}i_r i}_{j_{r-1}j_r j} h^{\ \alpha}_{i_{r-1}j_{r-1}} h^{\ \beta}_{i_r j_r} V_\beta + T_r{}^{i}_{j} V_\alpha\right) h^{\alpha}_{ijk} V_k\\
&\quad = \sum_{i,j,k,\alpha} (T_r{}^{i}_{j} h^{\alpha}_{ij})_{,k} V_k V_\alpha\\
&\quad = 0.
\end{aligned}
\tag{6.11}
$$

We have by use of (6.2), (6.1) and (3.8)

$$
\begin{aligned}
&\sum_{i,j,\alpha}\left(\frac{1}{r-1}\sum_{\substack{i_{r-1}i_r\\ j_{r-1}j_r\\ \beta}} T_{r-2}{}^{i_{r-1}i_r i}_{j_{r-1}j_r j} h^{\ \alpha}_{i_{r-1}j_{r-1}} h^{\ \beta}_{i_r j_r} V_\beta + T_r{}^{i}_{j} V_\alpha\right) \delta_{ij} V_\alpha \cdot c\\
&\quad = c\cdot\frac{n-r}{r-1}\sum_{\substack{i_{r-1}i_r\\ j_{r-1}j_r\\ \alpha,\beta}} T_{r-2}{}^{i_{r-1}i_r}_{j_{r-1}j_r} h^{\ \alpha}_{i_{r-1}j_{r-1}} h^{\ \beta}_{i_r j_r} V_\alpha V_\beta + c\cdot(n-r)\sum_{\alpha} S_r {V_\alpha}^2\\
&\quad = c\cdot(n-r)\Big(\sum_{\substack{i,j\\ \alpha,\beta}} T_{r-1}{}^{\alpha}{}_{ij} h^{\beta}_{ij} V_\alpha V_\beta + \sum_{\alpha} S_r {V_\alpha}^2\Big)
\end{aligned}
\tag{6.12}
$$

We have by use of (6.3) and (6.6)

$$
\sum_{i,j,\alpha}\left(\frac{1}{r-1}\sum_{\substack{i_{r-1}i_r\\ j_{r-1}j_r\\ \beta}} T_{r-2}{}^{i_{r-1}i_r i}_{j_{r-1}j_r j} h^{\ \alpha}_{i_{r-1}j_{r-1}} h^{\ \beta}_{i_r j_r} V_\beta + T_r{}^{i}_{j} V_\alpha\right) h^{\alpha}_{ik} h^{\gamma}_{kj} V_\gamma
$$

$$= \sum_{i,j,\alpha} \left(\frac{1}{r-1} \sum_{\substack{i_{r-1} i_r \\ j_{r-1} j_r \\ \beta}} T_{r-2}{}^{i_{r-1} i_r i}_{j_{r-1} j_r j} h^{\ \alpha}_{i_{r-1} j_{r-1}} h^{\alpha}_{ik} h^{\ \beta}_{i_r j_r} h^{\gamma}_{kj} V_\beta V_\gamma + \sum_{k,\gamma} T_{r}{}^{i}_{j} h^{\alpha}_{ik} h^{\gamma}_{kj} V_\alpha V_\gamma \right)$$

$$= \sum_{\beta,\gamma} \sum_{\substack{i_r j_r \\ jk}} (-T_{r}{}^{i_r k}_{j_r j} + \delta^k_j T_{r}{}^{i_r}_{j_r} - \delta^k_{j_r} T_{r}{}^{i_r}_{j}) h^{\ \beta}_{i_r j_r} h^{\gamma}_{kj} V_\beta V_\gamma + \sum_{\alpha,\beta} \sum_{i,j,k} T_{r}{}^{i}_{j} h^{\alpha}_{ik} h^{\beta}_{kj} V_\alpha V_\beta$$

$$= \sum_{\alpha,\beta} (-(r+1) \sum_{j,k} T_{r+1}{}^{\beta}_{kj} h^{\alpha}_{kj} V_\beta V_\alpha + \sum_{i_r j_r} T_{r}{}^{i_r}_{j_r} h^{\ \beta}_{i_r j_r} n H^\alpha V_\alpha V_\beta)$$

$$= -(r+1) \sum_{\alpha,\beta} \sum_{i,j} T_{r+1}{}^{\alpha}_{ij} h^{\beta}_{ij} V_\alpha V_\beta. \tag{6.13}$$

Putting (6.10), (6.11), (6.12) and (6.13) into (6.9), we get (6.4) from (6.8). Thus we complete the proof of Theorem 6.1.

Definition 6.1 Assume that r is even and $r \in \{0, 1, \ldots, n-1\}$, and let M be an n-dimensional compact r-minimal submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. If $J''_r(0) \geq 0$ for arbitrary variations, we call M to be stable.

7 Proofs of Theorem 1.4 and Theorem 1.5

Proof of Theorem 1.4 Now let $x \colon M \to S^{n+p} \hookrightarrow R^{n+p+1}$ be an n-dimensional compact submanifold of an $(n+p)$-dimensional unit sphere S^{n+p} in R^{n+p+1}. Suppose $\{e_i\}$ is a local orthonormal basis for TM and $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of $x \colon M \to S^{n+p}$. Let $E_1, \ldots, E_{n+p+1}$ be a canonical orthonormal basis of R^{n+p+1}. For a fixed $E_A \in R^{n+p+1}$, $1 \leq A \leq n+p+1$, we define

$$f_A = \langle x, E_A \rangle \tag{7.1}$$

$$g_{\alpha A} = \langle e_\alpha, E_A \rangle \tag{7.2}$$

for a normal vector e_α of the immersion $x \colon M \to S^{n+p}$, $n+1 \leq \alpha \leq n+p$.

Since x is contained in a unit sphere S^{n+p} in R^{n+p+1}, then

$$\begin{cases} \sum_{A=1}^{n+p+1} f_A^2 = \sum_A \langle x, E_A \rangle \langle x, E_A \rangle = \|x\|^2 = 1, \\ \sum_A f_A g_{\alpha A} = \sum_A \langle x, E_A \rangle \langle e_\alpha, E_A \rangle = \langle x, e_\alpha \rangle = 0, \\ \sum_A g_{\alpha A} g_{\beta A} = \sum_A \langle e_\alpha, E_A \rangle \langle e_\beta, E_A \rangle = \langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta}, \\ \sum_A \sum_{\alpha=n+1}^{n+p} g_{\alpha A}^2 = \sum_A \sum_\alpha \langle e_\alpha, E_A \rangle \langle e_\alpha, E_A \rangle = \sum_\alpha \langle e_\alpha, e_\alpha \rangle = p \end{cases} \tag{7.3}$$

Let $I(V^\perp) = J''_r(0)$, we now choose a special normal variational vector field

$$V^\perp = V_A^\perp = \sum_\alpha V_\alpha e_\alpha = \sum_\alpha g_{\alpha A} e_\alpha, \tag{7.4}$$

for a fixed index A, $1 \leq A \leq n+p+1$.

From Theorem 6.1, we have the following calculation by use of $c = 1$, (4.15), (7.3), $\mathbf{S}_{r+1} \equiv 0$ and Lemma 6.1

$$\sum_A I(V_A^\perp) = -\sum_A \int_M \sum_\alpha \langle e_\alpha, E_A\rangle \left\{ \frac{1}{r-1} \sum_{\substack{i_{r-1}i_r i\\ j_{r-1}j_r j\\ \beta}} T_{r-2}{}^{i_{r-1}i_r i}_{j_{r-1}j_r j} h^{\alpha}_{i_{r-1}j_{r-1}} h^{\beta}_{i_r j_r} \cdot \right.$$

$$\cdot \langle e_\beta, E_A\rangle_{,ij} + \sum_{i,j} T_r{}^i_j \langle e_\alpha, E_A\rangle_{,ij} + (n-r)\left(S_r \langle e_\alpha, E_A\rangle + \sum_{i,j,\beta} T_{r-1}{}^{\alpha}_{ij} h^\beta_{ij} \langle e_\beta, E_A\rangle \right)$$

$$\left. -(r+1)\sum_{i,j,\beta} T_{r+1}{}^{\alpha}_{ij} h^\beta_{ij}\langle e_\beta, E_A\rangle \right\} dv$$

$$= -\int_M \left\{ \frac{1}{r-1} \sum_A \sum_{\substack{i_{r-1}i_r i\\ j_{r-1}j_r j\\ \alpha\beta}} T_{r-2}{}^{i_{r-1}i_r i}_{j_{r-1}j_r j} h^{\alpha}_{i_{r-1}j_{r-1}} h^{\beta}_{i_r j_r} \langle e_\alpha, E_A\rangle \left(-\sum_k h^\beta_{ikj}\langle e_k, E_A\rangle \right.\right.$$

$$\left. -\sum_{k,\gamma} h^\beta_{ik} h^\gamma_{kj} \langle e_\gamma, E_A\rangle + h^\beta_{ij}\langle x, E_A\rangle \right) + \sum_A \sum_{i,j,\alpha} T_r{}^i_j \langle e_\alpha, E_A\rangle \left(-\sum_k h^\alpha_{ikj}\langle e_k, E_A\rangle \right.$$

$$\left. -\sum_{k,\gamma} h^\alpha_{ik} h^\gamma_{kj}\langle e_\gamma, E_A\rangle + h^\alpha_{ij}\langle x, E_A\rangle \right) + (n-r)\left(pS_r + \sum_{i,j,\alpha} T_{r-1}{}^{\alpha}_{ij} h^\alpha_{ij} \right)$$

$$\left. -(r+1)\sum_{i,j,\alpha} T_{r+1}{}^{\alpha}_{ij} h^\alpha_{ij} \right\} dv$$

$$= -\int_M \left\{ -\frac{1}{r-1} \sum_{\substack{i_{r-1}i_r i\\ j_{r-1}j_r j\\ \alpha\beta}} T_{r-2}{}^{i_{r-1}i_r i}_{j_{r-1}j_r j} h^{\alpha}_{i_{r-1}j_{r-1}} h^\alpha_{kj} h^{\beta}_{i_r j_r} h^\beta_{ik} - \sum_{i,j,k,\alpha} T_r{}^i_j h^\alpha_{ik} h^\alpha_{kj} \right.$$

$$\left. + (n-r)(pS_r + rS_r) - (r+2)(r+1)S_{r+2} \right\} dv$$

$$= -\int_M \left\{ \sum_{\substack{i_r i\\ j_r k\\ \beta}} (T_r{}^{i_r i}_{j_r k} - \delta^i_k T_r{}^{i_r}_{j_r} + \delta^i_{j_r} T_r{}^{i_r}_{k}) h^\beta_{i_r j_r} h^\beta_{ik} - \sum_{i,j,k,\alpha} T_r{}^i_j h^\alpha_{ik} h^\alpha_{kj} \right.$$

$$
\left. + (n-r)(p+r)S_r - (r+2)(r+1)S_{r+2} \right\} dv
$$

$$
\begin{aligned}
&= - \int_M \{(r+2)(r+1)S_{r+2} + (n-r)(p+r)S_r - (r+2)(r+1)S_{r+2}\} dv \\
&= - (n-r)(p+r) \int_M S_r dv
\end{aligned}
$$

When $S_r > 0$, we have $I(V^{\perp}) = J_r''(0) < 0$, i.e. if S_r is positive, there exists no any compact without boundary stable r-minimal submanifold in S^{n+p}, this completes the proof of Theorem 1.4.

Proof of Theorem 1.5 Let M be an n-dimensional compact r-minimal hypersurface in S^{n+1} with $S_r \geq 0$. Suppose M is stable, that is to say, $I(V^{\perp}) = J_r''(0) \geq 0$ for arbitrary variations. Choosing a special normal variational vector field $V = V_A^{\perp} = g_{n+1A}e_{n+1}$, we have above

$$
\sum_A I(V_A^{\perp}) = -(n-r)(r+1) \int_M S_r dv \geq 0. \tag{7.5}
$$

By assumption $S_r \geq 0$, we have

$$
S_r = 0 \tag{7.6}
$$

on M. On the other hand, by assumption condition that M is r-minimal, then

$$
S_{r+1} = 0. \tag{7.7}
$$

We have the following algebraic inequalities (see [14])

$$
H_r H_{r+2} \leq H_{r+1}^2 \tag{7.8}
$$

and equality holds in above inequality if and only if $k_1 = \cdots = k_n$.

From (7.6) and (7.7), we know that the equality in (7.8) holds, thus $k_1 = \cdots = k_n = 0$ on M, we conclude that M is a geodesic sphere.

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References

1. Alencar, H., do Carmo, M., Colares, A.G.: Stable hypersurfaces with constant scalar curvature. Math. Z. **213**, 117–131 (1993)
2. Alencar, H., do Carmo, M., Santos, W.: A gap theorem for hypersurfaces of the sphere with constant scalar curvature one. Comment. Math. Helv. **77**, 549–562 (2002)
3. Alencar, H., do Carmo, M., Elbert, M.F.: Stability of hypersurfaces with vanishing r-mean curvatures in Euclidean spaces. J. Reine Angew. Math. **554**, 201–216 (2003)
4. Alencar, H., Rosenberg, H., Santos, W.: On the Gauss map of hypersurfaces with constant scalar curvature in spheres. Proc. Amer. Math. Soc. **132**, 3731–3739 (2004)

5. Barbosa, J.L.M., Colares, A.G.: Stability of hypersurfaces with constant r-mean curvature. Ann. Global Anal. Geom. **15**, 277–297 (1997)
6. Barbosa, J.L.M., do Carmo, M.: On stability of cones in R^{n+1} with zero scalar curvature. Ann. Global Anal. Geom. **28**, 107–127 (2005)
7. Barbosa, J.L., do Carmo, M., Eschenburg, J.: Stability of hypersurfaces of constant mean curvature in Riemannian manifolds. Math. Z. **197**, 123–138 (1988)
8. Cheng, S.Y., Yau, S.T.: Hypersurfaces with constant scalar curvature. Math. Ann. **225**, 195–204 (1977)
9. Chern, S.S.: Minimal Submanifolds in a Riemannian Manifold (mimeographed). University of Kansas, Lawrence, (1968).
10. Do Carmo, M., Elbert, M.F.: On stable complete hypersurfaces with vanishing r-mean curvature. Tohoku Math. J. **56**(2), 155–162 (2004)
11. Grosjean, J.F.: Upper bounds for the first eigenvalue of the Laplacian on compact submanifolds. Pacific J. Math. **206**, 93–112 (2002)
12. Guo, Z., Li, H.: A variational problem for submanifolds in a sphere. To appear in Monatsh. Mat. (2007).
13. Guo, Z., Li, H., Wang, C.P.: The second variational formula for Willmore submanifolds in S^n. Results in Math. **40**, 205–225 (2001)
14. Hardy, G.H., Littlewood, J.E., Polya, G.: Inequalities. Cambridge University Press (1989)
15. Hu, Z.J., Li, H.: Willmore submanifolds in Riemannian manifolds. In: Proceedings of the Workshop, Contem. Geom. and Related Topics, pp. 251–275. World Scientific, May 2002
16. Li, A.M.: A class of variational problems and integral formula in Riemannian manifold. Chinese Acta Math. Sinica **28**(2), 145–153 (1985)
17. Li, H.: Hypersurfaces with constant scalar curvature in space forms Math. Ann. **305**, 665–672 (1996)
18. Li, H.: Global rigidity theorems of hypersurface. Ark. Math. **35**, 327–351 (1997)
19. Li, H.: Willmore hypersurfaces in a sphere. Asian J. Math. **5**, 365–378 (2001)
20. Li, H.: Willmore submanifolds in a sphere. Math. Res. Lett. **9**, 771–790 (2002)
21. Reilly, R.: Variation properties of functions of the mean curvatures for hypersurfaces in space forms. J. Differential. Geom. **8**, 465–477 (1973)
22. Reilly, R.: On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space. Comment. Math. Helvetici. **52**, 525–533 (1977)
23. Rosenberg, H.: Hypersurfaces of constant curvature in space forms. Bull. Soc. Math. **117**, 211–239 (1993)
24. Simons, J.: Minimal varieties in Riemmannian manifolds. Ann. Math. 2nd Ser. **88**(1), 62–105 (1968)
25. Takahashi, T.: Minimal immersions of Riemannian manifolds. J. Math. Soc. Japan **18**, 380–385 (1966)