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SCALAR CURVATURE, KILLING VECTOR FIELDS AND HARMONIC ONE-FORMS ON COMPACT RIEMANNIAN MANIFOLDS

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Dedicated to Professor Zhenzu Sun on the occasion of his 70th birthday.

Abstract

It is well known that no non-trivial Killing vector field exists on a compact Riemannian manifold of negative Ricci curvature; analogously, no non-trivial harmonic one-form exists on a compact manifold of positive Ricci curvature. One can consider the following, more general, problem. By reducing the assumption on the Ricci curvature to one on the scalar curvature, such vanishing theorems cannot hold in general. This raises the question: "What information can we obtain from the existence of non-trivial Killing vector fields (or, respectively, harmonic one-forms)?" This paper gives answers to this problem; the results obtained are optimal.

1. *Introduction*

A vector field V on a Riemannian manifold (M, g) is *Killing* if the Lie derivative of the metric with respect to V vanishes as follows:

$$L_V g = 0. \tag{1.1}$$

This is equivalent to the fact that the one-parameter group of diffeomorphisms generated by V consists of isometries. Therefore, the space of the non-trivial Killing vector fields for (M, g) in some sense measures the size of the isometry group of (M, g).

Using the technique that now bears his name, Bochner showed that when (M, g) is compact and has negative Ricci curvature, every Killing vector field must vanish; see [**2**]. In fact, Bochner's results hold for manifolds with quasinegative Ricci curvature, which means that the Ricci curvature is nonpositive on M and negative somewhere, for at least one point (see [**8**, **9**]). We also note that Bochner's result was extended by Yano [**10**] to include conformal vector fields. As is well known, the existence of non-trivial closed conformal vector fields also imposes many restrictions on a compact Riemannian manifold (see [**7**]).

On the other hand, for (M, g) with (quasi)positive Ricci curvature, a similar vanishing theorem holds for harmonic one-forms. More precisely, Bochner showed that when a compact Riemannian (M, g) has quasipositive Ricci curvature, every harmonic one-form must vanish [**2**, **8**, **9**].

Equivalently, the above results imply that if a Riemannian manifold is of negative scalar curvature and has nontrivial Killing vector fields, then the Ricci curvature cannot be negative everywhere. Likewise, if a Riemannian manifold is of positive

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scalar curvature and has nontrivial harmonic one-forms, then the Ricci curvature cannot be positive everywhere.

These facts give rise to a natural question: "If the scalar curvature R is less than 0, what restrictions are imposed by the existence of non-trivial Killing vector fields? Similarly, if the scalar curvature R is greater than 0, what restrictions are imposed by the existence of non-trivial harmonic one-forms?" This paper is motivated by these questions. We will answer them by proving the following optimal results.

THEOREM 1. *Let (M, g) be an n-dimensional compact oriented Riemannian manifold with scalar curvature $R < 0$. If there exists a non-trivial Killing vector field on (M, g), then we have*

$$\int_M \frac{1}{R}\left(|E|^2 - \frac{R^2}{n(n-1)}\right) d\,\mathrm{vol} \leqslant 0, \tag{1.2}$$

where E denotes the trace-free Ricci tensor. Moreover, equality is attained in (1.2) *if and only if R is constant and the Riemannian universal cover of (M, g) is isometric to a Riemannian product $\mathbb{R} \times N^{n-1}$ for some Einstein manifold N^{n-1} with constant scalar curvature R.*

THEOREM 2. *Let (M, g) be an n-dimensional compact oriented Riemannian manifold with scalar curvature $R > 0$. If the first de Rham cohomology group $H^1(M, \mathbb{R}) \neq \{0\}$, then every non-trivial harmonic one-form ω will satisfy*

$$\int_M \frac{1}{R}\left(\frac{R^2}{n(n-1)} - |E|^2\right) |\omega|^{(n-2)/(n-1)} d\,\mathrm{vol} \leqslant 0. \tag{1.3}$$

Moreover, equality is attained in (1.3) *if and only if R is constant and the Riemannian universal cover of (M, g) is isometric to a Riemannian product $\mathbb{R} \times N^{n-1}$ for some Einstein manifold N^{n-1} with constant scalar curvature R.*

REMARK 1.1. According to the Hodge theorem (see [**3**, p.84]), which states that the space of harmonic one-forms on (M, g) is isomorphic to $H^1(M, \mathbb{R})$, the topological condition $H^1(M, \mathbb{R}) \neq \{0\}$ is equivalent to (M, g) possessing non-trivial harmonic one-forms.

REMARK 1.2. The proofs of Theorems 1 and 2 are actually a refinement of the Bochner method, used to prove classical vanishing theorems. The ideas used in the refinement have been enlightened by Gursky's two papers [**4**, **5**], in which the author is concerned with conformal geometry for four-dimensional Riemannian manifolds.

We organize the paper as follows. In Section 2, we first review the related facts regarding Riemannian manifolds, using the method of moving frames. Then we state an algebraic lemma that turns out to be crucial for our purposes; we omit the proof of the lemma because it is standard. In Section 3, we derive the Weitzenböck formulas for Killing vector fields and harmonic one-forms, respectively, and then we present some useful lemmas. We finally give the proofs of Theorems 1 and 2, in Sections 4 and 5, respectively.

2. *Preliminaries*

Let (M,g) be an n-dimensional Riemannian manifold. Let $\{e_1,\dots,e_n\}$ be a local orthonormal basis of TM, adapted to the Riemannian metric g, and let $\{\theta_1,\dots,\theta_n\}$ be its dual basis. Let $\{\theta_{ij}\}$ be the connection forms of (M,g), which are defined by the structure equations

$$d\theta_i=\sum_j\theta_{ij}\wedge\theta_j,\qquad \theta_{ij}+\theta_{ji}=0, \tag{2.1}$$

$$d\theta_{ij}-\sum_k\theta_{ik}\wedge\theta_{kj}=-\frac{1}{2}\sum_{k,l}R_{ijkl}\theta_k\wedge\theta_l, \tag{2.2}$$

where R_{ijkl} are the components of the Riemannian curvature tensor of (M,g). Here and later in this paper, we adopt the Einstein convention: the ranges of indices are given by $i,j,k,l,m,\ldots=1,2,\ldots,n$.

The Ricci curvature tensor R_{ij} and the scalar curvature R of (M,g) are defined respectively by

$$R_{ij}=\sum_k R_{ikjk}\quad\text{and}\quad R=\sum_i R_{ii}. \tag{2.3}$$

We denote by ∇ the covariant differential operator of (M,g). For a vector field $V=\sum_i V_ie_i$ on (M,g), we define the covariant derivative $V_{i,j}$ by

$$\nabla V=\sum_{i,j}V_{i,j}\theta_j\otimes e_i$$

or, equivalently,

$$\sum_j V_{i,j}\theta_j=dV_i+\sum_j V_j\theta_{ji}. \tag{2.4}$$

Similarly, we can define the second covariant derivative $V_{i,jk}$ by

$$\sum_k V_{i,jk}\theta_k=dV_{i,j}+\sum_k V_{k,j}\theta_{ki}+\sum_k V_{i,k}\theta_{kj}. \tag{2.5}$$

Exterior derivation of (2.4) gives the Ricci identity

$$V_{i,kl}=V_{i,jk}+\sum_j V_jR_{jikl}. \tag{2.6}$$

Note that $L_Vg=0$ is equivalent to $g(\nabla_XV,Y)+g(X,\nabla_YV)=0$ for any $X,Y\in TM$. Thus the fact that $V=\sum_i V_ie_i$ is a Killing vector field is equivalent to having

$$V_{i,j}+V_{j,i}=0,\quad\text{for all } i,j. \tag{2.7}$$

In the same way, for a one-form $\omega=\sum_i\omega_i\theta_i$ on (M,g), the first and second covariant derivatives $\omega_{i,j}$ and $\omega_{i,jk}$ are defined, respectively, by

$$\sum_j\omega_{i,j}\theta_j=d\omega_i+\sum_j\omega_j\theta_{ji}, \tag{2.8}$$

$$\sum_k\omega_{i,jk}\theta_k=d\omega_{i,j}+\sum_k\omega_{k,j}\theta_{ki}+\sum_k\omega_{i,k}\theta_{kj}. \tag{2.9}$$

Exterior derivation of (2.8) gives the Ricci identity

$$\omega_{i,jk}=\omega_{i,kj}+\sum_l\omega_lR_{lijk}. \tag{2.10}$$

Recall that when M is compact, having a harmonic one-form ω is equivalent to having $d\omega = 0 = \delta\omega$, where $\delta\omega = \sum_i \omega_{i,i}$ denotes the codifferential of ω. It follows that ω is harmonic if and only if

$$\omega_{i,j} = \omega_{j,i}, \quad \text{for all } i, j, \quad \text{and} \quad \sum_i \omega_{i,i} = 0. \tag{2.11}$$

To prove Theorems 1 and 2, we need the following algebraic lemma, which can be proved by the standard method of Lagrange multipliers.

LEMMA 2.1. *Let $A = (a_{ij})_{n\times n}$ be a real symmetric matrix with $\sum_i a_{ii} = 0$ and $x_1, \dots, x_n \in \mathbb{R}$. Then*

$$-\sqrt{\frac{n-1}{n}\sum_{i,j} a_{ij}^2}\Big(\sum_i x_i^2\Big) \leqslant \sum_{i,j} a_{ij}x_i x_j \leqslant \sqrt{\frac{n-1}{n}\sum_{i,j} a_{ij}^2}\Big(\sum_i x_i^2\Big). \tag{2.12}$$

Moreover, when $\sum_i x_i^2 \neq 0$, one of the equalities in (2.12) *is attained if and only if there exists an orthogonal $n \times n$ matrix T such that*

$$TAT^{-1} = \begin{pmatrix} (n-1)\lambda & & & \\ & -\lambda & & \\ & & \ddots & \\ & & & -\lambda \end{pmatrix}$$

and $(x_1, \dots, x_n)$ correspondingly takes the value $((n-1)\lambda, 0, \dots, 0)$, where $\lambda > 0$ holds if there is equality in the right-hand side of (2.12), *and $\lambda < 0$ holds if there is equality in the left-hand side of* (2.12).

3. *The Weitzenböck formula and some lemmas*

We first consider the case of Killing vector fields.

If $V = \sum_i V_i e_i$ is a Killing vector field on the n-dimensional Riemannian manifold, then from (2.5) and (2.7), we have

$$\begin{aligned}\sum_{j,k} V_{j,jk}\theta_k &= \sum_j \Big(dV_{j,j} + \sum_k V_{k,j}\theta_{kj} + \sum_k V_{j,k}\theta_{kj}\Big)\\ &= d\Big(\sum_j V_{j,j}\Big) + \sum_{j,k}(V_{k,j} + V_{j,k})\theta_{kj}\\ &= 0,\end{aligned}$$

and thus

$$\sum_j V_{j,jk} = 0, \quad \text{for all } k. \tag{3.1}$$

From (2.6), (2.7) and (3.1) we obtain the following Weitzenböck formula:

$$\begin{aligned}\frac{1}{2}\Delta|V|^2 &= \sum_{i,j} V_{i,j}^2 + \sum_{i,j} V_i V_{i,jj} = |\nabla V|^2 - \sum_{i,j} V_i V_{j,ij}\\ &= |\nabla V|^2 - \sum_{i,j} V_i\Big(V_{j,ji} + \sum_k V_k R_{kjij}\Big)\\ &= |\nabla V|^2 - \sum_{i,j} V_i V_j R_{ij}.\end{aligned} \tag{3.2}$$

LEMMA 3.1. *Let $V = \sum_i V_i e_i$ be a Killing vector field on the n-dimensional Riemannian manifold (M, g). Then we have*

$$\sum_i V_i^2 \sum_{i,j} V_{i,j}^2 \geqslant 2 \sum_j \Big(\sum_i V_i V_{i,j} \Big)^2. \tag{3.3}$$

Proof. It suffices to prove (3.3) for any fixed point $p \in \Omega_0 := \{x \in M \mid V(x) \neq 0\}$. Note that on Ω_0, (3.3) is equivalent to

$$|\nabla V|^2 \geqslant 2 \big|\nabla |V|\big|^2. \tag{3.4}$$

Around p, we choose $\{e_i\}$ such that $V(p) = V_1(p) e_1(p)$; that is, $V_2 = \ldots = V_n = 0$ at p. From (2.7), we have

$$V_{1,1} = \ldots = V_{n,n} = 0; \qquad V_{1,j} = -V_{j,1}, \quad 2 \leqslant j \leqslant n.$$

Then at p we have

$$\begin{aligned} 2 \sum_j \Big(\sum_i V_i V_{i,j} \Big)^2 &= 2 V_1^2 \sum_j V_{1,j}^2 \\ &= 2 V_1^2 \sum_{j=2}^{n} V_{1,j}^2 \\ &\leqslant V_1^2 \sum_{i,j} V_{i,j}^2 \\ &= \sum_i V_i^2 \sum_{i,j} V_{i,j}^2. \end{aligned}$$

This proves (3.3), or equivalently proves (3.4) on Ω_0.

REMARK 3.1. In [**5**], Gursky observed that (3.3) still holds for every conformal vector field V that is defined to satisfy

$$V_{i,j} + V_{j,i} = f \delta_{ij}, \quad \text{for all } i, j,$$

for some function $f \in C^\infty(M)$.

For the Killing vector field $V = \sum_i V_i e_i$, we see that (2.6), (2.7) and (3.1) imply that

$$\Delta V_i = \sum_j V_{i,jj} = -\sum_j V_j R_{ij}, \qquad 1 \leqslant i \leqslant n. \tag{3.5}$$

From (3.5) and the unique continuation result of Kazdan [**6**], we know that, for a non-trivial Killing vector field V on the compact Riemannian manifold (M, g), the set $M \setminus \Omega_0$ is of measure zero. Combining this fact with (3.2) and (3.4), we have proved the following lemma.

LEMMA 3.2. *Let V be a non-trivial Killing vector field on a compact Riemannian manifold (M, g). Then*

$$\frac{1}{2} \Delta |V|^2 \geqslant 2 \big|\nabla |V|\big|^2 - \sum_{i,j} V_i V_j R_{ij} \tag{3.6}$$

holds on M in the sense of distributions.

The following lemma was first observed by Gursky in [**5**] for a conformal vector field. Here, for completeness, we include a new proof that clarifies Gursky's statement in [**5**].

LEMMA 3.3. *Let V be a non-trivial Killing vector field on the compact Riemannian manifold (M, g). For small $\epsilon > 0$, define $\Omega_\epsilon = \{x \in M \mid |V| \geqslant \epsilon\}$ and a function*

$$f_\epsilon(x) = \begin{cases} |V|(x), & \text{if } \ x \in \Omega_\epsilon, \\ \epsilon, & \text{if } \ x \notin \Omega_\epsilon; \end{cases}$$

then we have

$$\int_M -f_\epsilon^{-2} \sum_{i,j} V_i V_j R_{ij} \leqslant 0. \tag{3.7}$$

Proof. Multiplying both sides of (3.6) by f_ϵ^{-2} and then integrating over M, we obtain

$$\begin{aligned} -\int_M f_\epsilon^{-2} \sum_{i,j} V_i V_j R_{ij} &\leqslant \frac{1}{2} \int_M f_\epsilon^{-2} \Delta |V|^2 - 2 \int_M f_\epsilon^{-2} \big|\nabla |V|\big|^2 \\ &\leqslant \frac{1}{2} \int_M f_\epsilon^{-2} \Delta |V|^2 - 2 \int_{\Omega_\epsilon} |V|^{-2} \big|\nabla |V|\big|^2. \end{aligned} \tag{3.8}$$

If $\Omega_\epsilon \neq M$ for small $\epsilon > 0$, then $\partial \Omega_\epsilon$ locates in M as a closed hypersurface, which may be not connected, and $M = \Omega_\epsilon \cup \Omega_\epsilon^c$. Note that at each point $p \in \partial \Omega_\epsilon$, the outer unit normal vector $\vec{n}_1$ of the hypersurface $\partial \Omega_\epsilon \hookrightarrow \Omega_\epsilon$, and the outer unit normal vector $\vec{n}_2$ of $\partial \overline{\Omega_\epsilon^c} \hookrightarrow \overline{\Omega_\epsilon^c}$, differ only by a sign. Then, integrating by parts and using the fact that $f_\epsilon\big|_{\Omega_\epsilon^c} = \epsilon$ and $f_\epsilon\big|_{\Omega_\epsilon} = |V|$, we find that

$$\begin{aligned} \frac{1}{2} \int_M f_\epsilon^{-2} \Delta |V|^2 &= \frac{1}{2} \int_{\Omega_\epsilon} f_\epsilon^{-2} \Delta |V|^2 + \frac{1}{2} \int_{\Omega_\epsilon^c} f_\epsilon^{-2} \Delta |V|^2 \\ &= 2 \int_{\Omega_\epsilon} |V|^{-2} \big|\nabla |V|\big|^2 + \int_{\Omega_\epsilon} \Delta \log |V| + \frac{1}{2\epsilon^2} \int_{\Omega_\epsilon^c} \Delta |V|^2 \\ &= 2 \int_{\Omega_\epsilon} |V|^{-2} \big|\nabla |V|\big|^2 - \int_{\partial \Omega_\epsilon} \nabla \log |V| \cdot \vec{n}_1 - \frac{1}{2\epsilon^2} \int_{\partial \Omega_\epsilon^c} \nabla |V|^2 \cdot \vec{n}_2 \\ &= 2 \int_{\Omega_\epsilon} |V|^{-2} \big|\nabla |V|\big|^2. \end{aligned} \tag{3.9}$$

If $\Omega_\epsilon = M$ for small $\epsilon > 0$, it is clear that (3.9) still holds. Now (3.7) follows from (3.8) and (3.9). This proves Lemma 3.3.

To prove Theorem 2, we need to apply the above discussion for Killing vector fields to harmonic one-forms. Let $\omega = \sum_i \omega_i \theta_i$ be a harmonic one-form on the Riemannian manifold (M, g). Then (2.9) and (2.11) imply that

$$\begin{aligned} \sum_{j,k} \omega_{j,jk} \theta_k &= \sum_j d\omega_{j,j} + \sum_{j,k} \omega_{k,j} \theta_{kj} + \sum_{j,k} \omega_{j,k} \theta_{kj} \\ &= \sum_{j,k} (\omega_{k,j} + \omega_{j,k}) \theta_{kj} \\ &= 0; \end{aligned}$$

that is,

$$\sum_j \omega_{j,jk} = 0, \quad \text{for all } k. \tag{3.10}$$

From (2.10), (2.11) and (3.10) we obtain the following Weitzenböck formula:

$$\begin{aligned}\frac{1}{2}\Delta|\omega|^2 &= \sum_{i,j}\omega_{i,j}^2 + \sum_{i,j}\omega_i\omega_{i,jj}\\ &= |\nabla\omega|^2 + \sum_{i,j}\omega_i\omega_{j,ij}\\ &= |\nabla\omega|^2 + \sum_{i,j}\omega_i\Big(\omega_{j,ji} + \sum_k \omega_k R_{kjij}\Big)\\ &= |\nabla\omega|^2 + \sum_{i,j}\omega_i\omega_j R_{ij}. \end{aligned} \tag{3.11}$$

Lemma 3.4. *Let ω be a non-trivial harmonic one-form on an n-dimensional Riemannian manifold (M, g), and let $D_0 = \{x \in M \mid \omega(x) \neq 0\}$. Then, on D_0, ω satisfies*

$$|\nabla\omega|^2 \geqslant \frac{n}{n-1}\big|\nabla|\omega|\,\big|^2. \tag{3.12}$$

Proof. For $p \in D_0$ and $\omega = \sum_i \omega_i\theta_i$, inequality (3.12) is equivalent to

$$\sum_i \omega_i^2 \sum_{i,j}\omega_{i,j}^2 \geqslant \frac{n}{n-1}\sum_j\Big(\sum_i \omega_i\omega_{i,j}\Big)^2. \tag{3.13}$$

Around p, we choose $\{e_i\}$ such that $\omega_2 = \ldots = \omega_n = 0$ at p. This reduces (3.13) to the form

$$\omega_1^2\sum_{i,j}\omega_{i,j}^2 \geqslant \frac{n}{n-1}\omega_1^2\sum_j \omega_{1,j}^2. \tag{3.13$'$}$$

However, we can apply (2.11) to obtain

$$\begin{aligned}\sum_{i,j}\omega_{i,j}^2 &= \sum_j \omega_{1,j}^2 + \sum_{j\geqslant 2}\omega_{j,1}^2 + \sum_{i,j\geqslant 2}\omega_{i,j}^2\\ &\geqslant \sum_j \omega_{1,j}^2 + \sum_{j\geqslant 2}\omega_{j,1}^2 + \sum_{j\geqslant 2}\omega_{j,j}^2\\ &\geqslant \sum_j \omega_{1,j}^2 + \sum_{j\geqslant 2}\omega_{j,1}^2 + \frac{1}{n-1}\Big(\sum_{j\geqslant 2}\omega_{j,j}\Big)^2\\ &= \sum_j \omega_{1,j}^2 + \sum_{j\geqslant 2}\omega_{j,1}^2 + \frac{1}{n-1}\omega_{1,1}^2\\ &\geqslant \sum_j \omega_{1,j}^2 + \frac{1}{n-1}\sum_j \omega_{j,1}^2\\ &= \frac{n}{n-1}\sum_j \omega_{1,j}^2; \end{aligned} \tag{3.14}$$

then (3.13′) follows. This proves Lemma 3.4.

As in (3.5), for a harmonic one-form $\omega = \sum_i \omega_i\theta_i$, equations (2.10) and (3.10) give

$$\Delta\omega_i = \sum_j \omega_{i,jj} = \sum_j \left(\omega_{j,ji} + \sum_k \omega_k R_{kjij}\right) = \sum_j \omega_j R_{ij}, \qquad 1 \leqslant i \leqslant n. \quad (3.15)$$

From (3.15) and the unique continuation result of Kazdan [**6**], we know that for a non-trivial harmonic one-form ω on the compact Riemannian manifold (M, g), the set $M \setminus D_0$ is of measure zero. Combining this fact with (3.11) and (3.12), we have proved the following lemma.

Lemma 3.5. *Let ω be a non-trivial harmonic one-form on a compact Riemannian manifold (M, g). Then*

$$\frac{1}{2}\Delta|\omega|^2 \geqslant \frac{n}{n-1}\big|\nabla|\omega|\,\big|^2 + \sum_{i,j}\omega_i\omega_j R_{ij} \quad (3.16)$$

holds on M in the sense of distributions.

As a counterpart of Lemma 3.3, we have the next lemma.

Lemma 3.6. *Let ω be a non-trivial harmonic one-form on the compact n-dimensional Riemannian manifold (M, g). For small $\epsilon > 0$, define $D_\epsilon = \{x \in M \mid |\omega| \geqslant \epsilon\}$, and a function*

$$h_\epsilon(x) = \begin{cases} |\omega|(x), & \text{if } x \in D_\epsilon, \\ \epsilon, & \text{if } x \notin D_\epsilon; \end{cases}$$

then we have

$$\int_M h_\epsilon^{-n/(n-1)} \sum_{i,j}\omega_i\omega_j R_{ij} \leqslant 0. \quad (3.17)$$

Proof. Multiplying both sides of (3.16) by $h_\epsilon^{-n/(n-1)}$ and then integrating over M, we obtain

$$\begin{aligned}\int_M h_\epsilon^{-n/(n-1)} \sum_{i,j}\omega_i\omega_j R_{ij} &\leqslant \frac{1}{2}\int_M h_\epsilon^{-n/(n-1)}\Delta|\omega|^2 - \frac{n}{n-1}\int_M h_\epsilon^{-n/(n-1)}\big|\nabla|\omega|\,\big|^2 \\ &\leqslant \frac{1}{2}\int_M h_\epsilon^{-n/(n-1)}\Delta|\omega|^2 - \frac{n}{n-1}\int_{D_\epsilon} |\omega|^{-n/(n-1)}\big|\nabla|\omega|\,\big|^2.\end{aligned} \quad (3.18)$$

By a similar argument to that used in proving (3.9), we can show, by using the facts that $h_\epsilon\big|_{D_\epsilon^c} = \epsilon$ and $h_\epsilon\big|_{D_\epsilon} = |\omega|$, that

$$\frac{1}{2}\int_M h_\epsilon^{-n/(n-1)}\Delta|\omega|^2 = \frac{n}{n-1}\int_{D_\epsilon} |\omega|^{-n/(n-1)}\big|\nabla|\omega|\,\big|^2. \quad (3.19)$$

Now (3.17) follows from (3.18) and (3.19). This proves Lemma 3.6.

4. *Proof of Theorem* 1

Let (M, g) be a compact, oriented n-dimensional Riemannian manifold with scalar curvature $R < 0$, and let $V = \sum_i V_i e_i$ be a non-trivial Killing vector field on (M, g). Denote by E the trace-free part of the Ricci tensor Ric; that is, $E_{ij} = R_{ij} - (R/n)\delta_{ij}$. Then, applying Lemma 2.1, we see that

$$-\sum_{i,j} V_i V_j R_{ij} = -\sum_{i,j} V_i V_j E_{ij} - \frac{R}{n}|V|^2$$
$$\geqslant -\sqrt{\frac{n-1}{n}}|E|\,|V|^2 - \frac{R}{n}|V|^2, \tag{4.1}$$

and the second equality holds at a point $p \in M$ with $V(p) \neq 0$ if and only if E can be diagonalized (by an appropriate choice of $\{e_i\}$) as

$$E = \begin{pmatrix} (n-1)\lambda & & & \\ & -\lambda & & \\ & & \ddots & \\ & & & -\lambda \end{pmatrix}; \tag{4.2}$$

correspondingly, $V_1 = (n-1)\lambda$ and $V_2 = \ldots = V_n = 0$ for some $\lambda > 0$.

From (4.1), we see that

$$-\sum_{i,j} V_i V_j R_{ij} \geqslant -\frac{n-1}{2R}\left[\left(|E| + \frac{R}{\sqrt{n(n-1)}}\right)^2 - \left(|E|^2 - \frac{R^2}{n(n-1)}\right)\right]|V|^2$$
$$\geqslant \frac{n-1}{2R}\left(|E|^2 - \frac{R^2}{n(n-1)}\right)|V|^2. \tag{4.3}$$

For brevity, we omit '$d\,$vol' from the integrals that follow. Substituting (4.3) into (3.7) gives

$$0 \geqslant \int_M \frac{n-1}{2R}\left(|E|^2 - \frac{R^2}{n(n-1)}\right)|V|^2 f_\epsilon^{-2}, \quad \text{for small } \epsilon > 0. \tag{4.4}$$

Note that

$$\int_M \frac{1}{R}\left(|E|^2 - \frac{R^2}{n(n-1)}\right)|V|^2 f_\epsilon^{-2}$$
$$= \int_M \frac{1}{R}\left(|E|^2 - \frac{R^2}{n(n-1)}\right) + \int_M \frac{1}{R}\left(|E|^2 - \frac{R^2}{n(n-1)}\right)\left(|V|^2 f_\epsilon^{-2} - 1\right)$$
$$= \int_M \frac{1}{R}\left(|E|^2 - \frac{R^2}{n(n-1)}\right) + \int_{M\setminus\Omega_\epsilon} \frac{1}{R}\left(|E|^2 - \frac{R^2}{n(n-1)}\right)\left(|V|^2 f_\epsilon^{-2} - 1\right)$$

and

$$-1 \leqslant |V|^2 f_\epsilon^{-2} - 1 \leqslant 0, \quad \text{on } M \setminus \Omega_\epsilon,$$
$$\lim_{\epsilon \to 0} \mathrm{Vol}\left(M \setminus \Omega_\epsilon\right) = \mathrm{Vol}\left(M \setminus \Omega_0\right) = 0;$$

from the compactness of M, we can take the limit as $\epsilon \to 0$ in (4.4) to conclude that

$$\int_M \frac{1}{R}\left(|E|^2 - \frac{R^2}{n(n-1)}\right) \leqslant 0. \tag{4.5}$$

If the equality holds in (4.5), then the integral version of the corresponding equality in (4.3) holds, and the equality in (4.1) must hold at each point of M.

Thus E can be diagonalized (at each point of M) as in (4.2); it satisfies

$$|E| \equiv -\frac{R}{\sqrt{n(n-1)}} \quad \text{on } M. \tag{4.6}$$

Furthermore, combining (4.2) with (4.6) gives $\lambda = -R/n(n-1)$. Then we must have $V_1 = -R/n$, $V_2 = \ldots = V_n = 0$ and

$$\mathrm{Ric} = (R_{ij}) = \begin{pmatrix} 0 & & & \\ & \frac{R}{n-1} & & \\ & & \ddots & \\ & & & \frac{R}{n-1} \end{pmatrix}. \tag{4.7}$$

This implies that $\sum_{i,j} V_i V_j R_{ij} = 0$, and thus from (3.2) we know that $|V| = -R/n$ is constant and $\nabla V = 0$; that is, V is in fact a parallel vector field on (M, g). Passing to the Riemannian universal cover $\tilde{M}$ of M, by the de Rham decomposition theorem $\tilde{M}$ splits, and by (4.7) we see that $(\tilde{M}, g)$ is the product of $\mathbb{R}$ and an Einstein manifold $\tilde{N}$ of Ricci curvature $R/(n-1) < 0$.

This completes the proof of Theorem 1.

5. *Proof of Theorem* 2

The procedure for this proof is completely parallel to that described in Section 4. Let (M, g) be a compact, oriented n-dimensional Riemannian manifold with scalar curvature $R > 0$, and let $\omega = \sum_i \omega_i \theta_i$ be a non-trivial harmonic one-form on M. By writing $E_{ij} = R_{ij} - (R/n)\delta_{ij}$, we have

$$\sum_{i,j} \omega_i \omega_j R_{ij} = \sum_{i,j} \omega_i \omega_j E_{ij} + \frac{R}{n}|\omega|^2. \tag{5.1}$$

Since $\sum_i E_{ii} = 0$ and $\omega \neq 0$, by applying Lemma 2.1 we obtain

$$\sum_{i,j} \omega_i \omega_j E_{ij} \geqslant -\sqrt{\frac{n-1}{n}}|E|\,|\omega|^2, \tag{5.2}$$

and equality holds at a point $p \in M$ with $\omega(p) \neq 0$ if and only if E can be diagonalized (by an appropriate choice of $\{e_i\}$) as

$$E = \begin{pmatrix} (n-1)\lambda & & & \\ & -\lambda & & \\ & & \ddots & \\ & & & -\lambda \end{pmatrix}; \tag{5.3}$$

correspondingly, $\omega_1 = (n-1)\lambda$ and $\omega_2 = \ldots = \omega_n = 0$ for some $\lambda < 0$.

From (5.1) and (5.2), we see that

$$\begin{aligned} \sum_{i,j} \omega_i \omega_j R_{ij} &\geqslant \Big(-\sqrt{\frac{n-1}{n}}|E| + \frac{R}{n} \Big)|\omega|^2 \\ &= \frac{n-1}{2R}\Big[\Big(|E| - \frac{R}{\sqrt{n(n-1)}} \Big)^2 - \Big(|E|^2 - \frac{R^2}{n(n-1)} \Big) \Big]|\omega|^2 \\ &\geqslant \frac{n-1}{2R}\Big(\frac{R^2}{n(n-1)} - |E|^2 \Big)|\omega|^2. \end{aligned} \tag{5.4}$$

Substituting this into (3.17) gives

$$0 \geqslant \int_M \frac{n-1}{2R}\left(\frac{R^2}{n(n-1)} - |E|^2\right)|\omega|^2 h_\epsilon^{-n/(n-1)}, \quad \text{for small } \epsilon > 0. \tag{5.5}$$

Taking the limit as $\epsilon \to 0$ and observing that $|\omega|^2 h_\epsilon^{-2} \to 1$ almost everywhere on M (or, alternatively, by the same argument as that used in deriving (4.5)), we conclude that

$$\int_M \frac{1}{R}\left(\frac{R^2}{n(n-1)} - |E|^2\right)|\omega|^{(n-2)/(n-1)} \leqslant 0. \tag{5.6}$$

If the equality holds in (5.6), then the equalities in (5.2) and (5.4) must hold at each point of M. Thus E can be diagonalized (at each point of M) as in (5.3), and it satisfies (by continuity)

$$|E| \equiv \frac{R}{\sqrt{n(n-1)}} \quad \text{on } M. \tag{5.7}$$

Furthermore, combining (5.3) with (5.7) gives $\lambda = -R/n(n-1)$. Then we must have

$$\mathrm{Ric} = (R_{ij}) = \begin{pmatrix} 0 & & & \\ & \frac{R}{n-1} & & \\ & & \ddots & \\ & & & \frac{R}{n-1} \end{pmatrix}; \tag{5.8}$$

also, $\omega_1 = -R/n$ and $\omega_2 = \ldots = \omega_n = 0$. This implies that

$$\sum_{i,j} \omega_i \omega_j R_{ij} = 0.$$

Thus from (3.11) we know that $|\omega| = R/n$ is constant and $\nabla\omega = 0$; that is, ω is in fact a parallel one-form on (M, g). Passing to the Riemannian universal cover $\tilde{M}$ of M, by the de Rham decomposition theorem $\tilde{M}$ splits, and by (5.8) $(\tilde{M}, g)$ is the product of $\mathbb{R}$ and an Einstein manifold $\tilde{N}$ of Ricci curvature $R/(n-1) > 0$.

This completes the proof of Theorem 2.

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