Math. Z. 245, 201–216 (2003)

Mathematische Zeitschrift

DOI: 10.1007/s00209-003-0512-4

Quantization of curvature for compact surfaces in $S^{n\star}$

Haizhong Li[1,2], **Udo Simon**[1]

[1] Institut für Mathematik, MA 8-3, Technische Universität Berlin, Strasse des 17. Juni 136, 10623 Berlin, Germany
(e-mail: hli@math.tu-berlin.de/simon@math.tu-berlin.de)

[2] Department of Mathematical Sciences, Tsinghua University, 100084, Beijing, People's Republic of China,
(e-mail: hli@math.tsinghua.edu.cn)

Received: 19 February 2002; in final form 25 September 2002 /
Published online 16 August 2003 – © Springer-Verlag 2003

Abstract. For minimal surfaces in spheres, there is a well known conjecture about the quantization of intrinsic curvature which has been solved only in special cases so far. We recall an intrinsic and an extrinsic version for the known results and extend them to compact non-minimal surfaces in spheres. In particular we discuss special classes like Willmore surfaces and surfaces with parallel mean curvature vector.

Mathematics Subject Classification (2000): 53C42, 53A10.

1. Introduction

In [3], E. Calabi considered minimal immersions of compact surfaces without boundary and with constant Gauss curvature K into $S^n(1)$. He gave a complete list of all such immersions and proved that the set of possible values of K is discrete, namely $K = K(s)$:

$$K(s) = \frac{2}{s(s+1)}, \quad s \in \mathbb{N}.$$

This led to the so called (see [14]])

Quantization conjecture (intrinsic version). *Let (M, g) be a compact surface minimally immersed into $S^n(1)$; denote by K the curvature of the Riemannian metric g. If*

⋆ H. Li is partially supported by a research fellowship of the Alexander von Humboldt Stiftung 2001/2002 and the Zhongdian grant of NSFC. U. Simon is partially supported by DFG 163/Si-7-2 and a Chinese–German research cooperation of NSFC and DFG.

$$K(s+1) \leq K \leq K(s)$$

for an $s \in \mathbb{N}$, *then either* $K = K(s)$ *or* $K = K(s+1)$, *and the immersion is one of Calabi's standard immersions.*

Here and in the following we consider compact surfaces without boundary. So far, this conjecture has been solved only in the cases $s = 1$ and $s = 2$ (see [8], [1], [7]); under additional assumptions there are many partial solutions for $s \geq 3$ (see e.g. [2], [5], [6], [11]], [12], [13]). There is another version of this quantization conjecture for the extrinsic curvature function, namely for the square of the length of the second fundamental form, $|II|^2$. For minimal surfaces in $S^n(1)$, both curvature functions are related as follows:

$$2K = 2 - |II|^2.$$

Thus, for Calabi's standard immersions, we have

$$2K(s) =: 2 - |II(s)|^2 \quad \text{and} \quad |II(s)|^2 = \frac{2(s-1)(s+2)}{s(s+1)}, \quad s \in \mathbb{N}.$$

From this we get:

Quantization conjecture (extrinsic version). *Let* (M, g) *be a compact surface minimally immersed into* $S^n(1)$. *If*

$$|II(s)|^2 \leq |II|^2 \leq |II(s+1)|^2$$

then $|II|^2 = const$ *and either* $|II|^2 = |II(s)|^2$ *or* $|II|^2 = |II(s+1)|^2$, *and the immersion is one of Calabi's standard immersions.*

For a minimal immersion as considered above, $K = K(s) = 1$ for $s = 1$ gives $|II|^2 = 0$, and the immersion is an equator in $S^3(1)$, while $K = K(s) = \frac{1}{3}$ for $s = 2$ gives $|II|^2 = \frac{4}{3}$ and the immersion is a Veronese surface in $S^4(1)$ which can be described as in Example 1 below.

In this paper, we drop the assumption on the minimality of the immersion and consider more general classes of surfaces. We use the following notation. Let $x : M \to S^n(1)$ be a surface in an n-dimensional unit sphere $S^n(1)$. Let e_α $(3 \leq \alpha \leq n)$ are local orthonormal normal vector fields of M in $S^n(1)$. Let h^α denote the second fundamental form with respect to e_α, h^α_{ij} its local components, $\vec{H}$ the mean curvature vector and H the mean curvature function of M; we have

$$|II|^2 = \sum_\alpha \sum_{i,j} (h^\alpha_{ij})^2, \quad \vec{H} = \sum_\alpha H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{2} \sum_k h^\alpha_{kk}, \quad H = |\vec{H}|.$$

Example 1. (*see* [4] *or* [8]). Veronese surface. Let (x, y, z) denote the canonical coordinate system in $\mathbb{R}^3$ and $u = (u_1, u_2, u_3, u_4, u_5)$ the canonical coordinates in $\mathbb{R}^5$. We consider the mapping defined by

$$u_1 = \frac{1}{\sqrt{3}} yz, \quad u_2 = \frac{1}{\sqrt{3}} xz, \quad u_3 = \frac{1}{\sqrt{3}} xy,$$

$$u_4 = \frac{1}{2\sqrt{3}}(x^2 - y^2), \quad u_5 = \frac{1}{6}(x^2 + y^2 - 2z^2),$$

where $x^2 + y^2 + z^2 = 3$. This defines an isometric immersion of $S^2(\sqrt{3})$ into $S^4(1)$. Two points (x, y, z) and $(-x, -y, -z)$ of $S^2(\sqrt{3})$ are mapped into the same point of $S^4(1)$. This real projective plane imbedded in $S^4(1)$ is called the *Veronese surface*. We know that the Veronese surface is a minimal surface in $S^4(1)$ (see [4] or [8]). As stated above, $|II|^2 = \frac{4}{3}$ and $K = \frac{1}{3}$.

For $s = 1$ and $n = 4$, the first result concerning the quantization of intrinsic curvature was proved by B. Lawson [8]; for $s = 1$ and arbitrary n, the intrinsic quantization result is the consequence of the following integral inequality for minimal immersions (see Benko-Kothe-Semmler-Simon [1] or Kozlowski-Simon [7]).

Theorem 1 (intrinsic version). *Let M be a compact minimal surface with Gauss curvature K in an n-dimensional unit sphere $S^n(1)$. Then we have*

$$\int_M (1 - K)(3K - 1)dv \leq 0. \tag{1.1}$$

In particular, if K is in a so called quantization interval

$$\frac{1}{3} \leq K \leq 1, \tag{1.2}$$

then either $K = 1$ and M is totally geodesic, or $K = \frac{1}{3}$, $n = 4$ and M is the Veronese surface given by Example 1.

Theorem 1 (extrinsic version). *Let M be a compact minimal surface in an n-dimensional unit sphere $S^n(1)$. Then we have*

$$\int_M |II|^2(4 - 3|II|^2)dv \leq 0. \tag{1.3}$$

In particular, if $|II|^2$ is in a so called quantization interval

$$0 \leq |II|^2 \leq \frac{4}{3} \tag{1.4}$$

then again, either M is totally geodesic, or M is the Veronese surface.

The equivalence of both versions follows from $2K = 2 - |II|^2$. For arbitrary dimension and codimension, the extrinsic version of the integral-inequality has been proved in [4]. In dimension 2, this version has been extended to Willmore surfaces in a recent paper of H. Li [9].

As already stated, in our paper we consider compact surfaces in $S^n(1)$ and extend the different versions of the quantization results; for this, we recall that, for arbitrary surfaces in $S^n(1)$, we have

$$2K = 2 + 4H^2 - |II|^2.$$

Each normal e_α induces principal curvatures $k_1(\alpha), k_2(\alpha)$; then

$$0 \leq \frac{1}{2}\sum_\alpha (k_1(\alpha) - k_2(\alpha))^2 = |II|^2 - 2H^2 =: \rho^2$$

where $\rho^2 = 2(1 + H^2 - K)$ is the integrand of the Willmore functional and $\rho^2(p) = 0$ exactly if p is umbilical, i.e. $h^\alpha = H^\alpha g$ for all α at p, where g is the first fundamental form. The function ρ^2 is an appropriate extrinsic curvature function to be considered on arbitrary surfaces in $S^n(1)$.

Our main tool for our quantization results below are different versions of a new integral inequality; the different versions are related to the foregoing intrinsic and extrinsic versions for minimal surfaces in spheres. The proofs in sections 3 and 4 depend on a subtle construction of a traceless, totally symmetric (0,3) tensor field W in Lemma 2.1. It is this construction of W which guarantees optimal results in our quantization theorems, where optimal means that we – with one exception - are able to discuss the equality at both boundaries of the quantization intervals considered.

Integral inequalities. *Let M be a compact surface in an n-dimensional unit sphere $S^n(1)$. Then*

$$0 \geq \int_M [2(1 - (K - H^2))(3(K - H^2) - 1)]dv + \int_M [H^2\rho^2 - \sum_{\alpha,i} (H^\alpha_{,i})^2]dv$$
$$= \int_M [\rho^2(2 - \frac{3}{2}\rho^2)]dv + \int_M [H^2\rho^2 - \sum_{\alpha,i} (H^\alpha_{,i})^2]dv.$$

Here the terms $H^\alpha_{,i}$ denote derivatives in local notation (s. Lemma 3.3).

In particular, if M is minimal, the integral inequalities reduce to the different versions (1.1) and (1.3), and we get the intrinsic quantization result from Theorem 1. As a first consequence we present a general result for compact surfaces.

Theorem 2. *Let M be a compact surface in an n-dimensional unit sphere $S^n(1)$. If M satisfies the integral inequality*

$$\int_M [H^2\rho^2 - \sum_{\alpha,i} (H^\alpha_{,i})^2]dv \geq 0 \tag{1.5.a}$$

and if additionally one of the following two (equivalent) inequalities (1.5.b) or (1.5.c) is satisfied at each point p,

$$\frac{1}{3} \leq K - H^2 \leq 1, \tag{1.5.b}$$

$$0 \leq \rho^2 \leq \frac{4}{3}, \tag{1.5.c}$$

then $H = const$, and either $n = 3$, $\rho^2 = 0$, $K = 1 + H^2$ and M is totally umbilical in $S^3(1)$, or $n = 4$, $\rho^2 = \frac{4}{3}$, $K = \frac{1}{3}$ and M is the Veronese surface given by Example 1.

The first condition, the integral inequality (1.5.a), is satisfied on special classes of surfaces (see section 5 below). On Willmore surfaces we have the case of equality; this class contains the class of minimal surfaces. Version (ii) of the following Theorem 3 has been stated and proved in [9]; later we will prove it as a consequence of a more general result.

Theorem 3. *Let M be a compact Willmore surface in an n-dimensional unit sphere $S^n(1)$. Then we have*

(i)

$$\int_M [1-(K-H^2)][3(K-H^2)-1]dv \le 0. \tag{1.6.a}$$

(ii) (see [9])

$$\int_M \rho^2(2-\frac{3}{2}\rho^2)dv \le 0. \tag{1.6.b}$$

In particular, if one of the two inequalities (1.5.b) or (1.5.c) above is satisfied, then $H = const$, and either $n = 3$, $\rho^2 = 0$, $K = 1 + H^2$ and M is totally umbilical in $S^3(1)$, or $n = 4$, $\rho^2 = \frac{4}{3}$, $K = \frac{1}{3}$ and M is the Veronese surface given by Example 1.

Remark. We would like to comment on the character of the conditions $(1.5.b-c)$. For this, the reader first should recall the following fact from hypersurface theory in space forms of constant curvature $\bar{K}$: Consider the curvature functions H_1, .. H_n, defined as (normed) elementary symmetric functions of the principal curvatures. Then it is a consequence of the generalized theorema egregium that the mean curvature $H := H_1$ is the only genuine extrinsic curvature invariant (see e.g. the introduction of [15]), while the other elementary symmetric curvature functions $H_r (r \ge 2)$ of the principal curvatures can be described (for odd order modulo sign) in terms of the metric and $\bar{K}$; the proof follows the lines of that of Theorem 5.3 in [10]. This fact explains the particular interest in the mean curvature and in relations between H as extrinsic and K as intrinsic curvature. For higher codimension, the mean curvature as well as the two functions $|II|^2$ and ρ^2 are the most important extrinsic curvature invariants. For minimal surfaces, the inequality (1.5.b) reduces to (1.2); but if $H \ne 0$, this inequality describes a quantitative control of the intrinsic Gauss curvature K in terms of the extrinsic mean curvature by

$$\frac{1}{3} + H^2 \le K \le 1 + H^2. \tag{1.5.b}$$

Like for minimal surfaces, the length of the interval for K is $\frac{2}{3}$ at each point; but as H might be a nonconstant function, the boundaries of the interval depend on the point. The above results suggest the study of a quantitative control of curvature functions on arbitrary surfaces.

2. Preliminaries

Let $x : M \to S^n(1)$ be a surface in an n-dimensional unit sphere. We choose an orthonormal basis $e_1, \cdots, e_n$ of $S^n(1)$ such that $\{e_1, e_2\}$ are tangent to $x(M)$ and $\{e_3, \cdots, e_n\}$ is a local frame in the normal bundle. Let $\{\omega_1, \omega_2\}$ be the dual forms of $\{e_1, e_2\}$. We use the following convention on the ranges of indices:

$$1 \le i, j, k, \cdots \le 2; \qquad 3 \le \alpha, \beta, \gamma, \cdots \le n.$$

Then we have the structure equations

$$dx = \sum_i \omega_i e_i, \tag{2.1}$$

$$de_i = \sum_j \omega_{ij} e_j + \sum_{\alpha, j} h^\alpha_{ij} \omega_j e_\alpha - \omega_i x, \tag{2.2}$$

$$de_\alpha = -\sum_{i,j} h^\alpha_{ij} \omega_j e_i + \sum_\beta \omega_{\alpha\beta} e_\beta, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.3}$$

The Gauss equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}), \tag{2.4}$$

this implies

$$R_{ik} = \delta_{ik} + 2\sum_\alpha H^\alpha h^\alpha_{ik} - \sum_{\alpha, j} h^\alpha_{ij} h^\alpha_{jk}, \tag{2.5}$$

$$2K = 2 + 4H^2 - |II|^2. \tag{2.6}$$

We consider the canonical embedding of $S^n(1)$ in $\mathbb{R}^{n+1}$ and describe $x(M)$ as surface in $\mathbb{R}^{n+1}$ in terms of its position vector, again denoted by x. $\mathbb{R}^{n+1}$ is equipped with an Euclidean structure, defined by the canonical scalar product $<,>: \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$. Then we have

$$< x, x >= 1, \quad < x, x_i >= 0. \tag{2.7}$$

Denote by Δ the Laplacian of the metric and, in local notation, covariant derivatives of x by $x_i, x_{ij}, x_{ijk}, x_{ijkl}$; then

$$x_{ij} = -\delta_{ij} x + \sum_\alpha h^\alpha_{ij} e_\alpha, \tag{2.8}$$

$$\Delta x = -2x + 2\vec{H}, \tag{2.9}$$

$$< \Delta x, \Delta x > = 4 + 4H^2, \quad H^2 := |\vec{H}|^2, \tag{2.10}$$

$$\sum_j < x_{ij}, x_{jk} > = \delta_{ik} + \sum_{j,\alpha} h^\alpha_{ij} h^\alpha_{jk}, \tag{2.11}$$

$$\sum_{i,j} < x_{ij}, x_{ij} > = 2 + |II|^2. \tag{2.12}$$

The vector valued Ricci identities for x read

$$x_{ijk} - x_{ikj} = \sum_l x_l R_{lijk}, \tag{2.13}$$

$$x_{ijkl} - x_{ijlk} = \sum_m x_{mj} R_{mikl} + \sum_m x_{im} R_{mjkl}, \tag{2.14}$$

where x_i, x_{ij}, x_{ijk} and x_{ijkl} satisfy

$$dx = \sum_i x_i \omega_i, \tag{2.15}$$

$$\sum_j x_{ij}\omega_j = dx_i + \sum_j x_j \omega_{ji}, \tag{2.16}$$

$$\sum_k x_{ijk}\omega_k = dx_{ij} + \sum_k x_{kj}\omega_{ki} + \sum_k x_{ik}\omega_{kj}, \tag{2.17}$$

$$\sum_l x_{ijkl}\omega_l = dx_{ijk} + \sum_l x_{ljk}\omega_{li} + \sum_l x_{ilk}\omega_{lj} + \sum_l x_{ijl}\omega_{lk}. \tag{2.18}$$

We have the following formula

$$\begin{aligned}\sum_k x_{ijkk} &= (\Delta x)_{ij} + \sum_m x_{mj} R_{mi} + \sum_m x_{im} R_{mj} + \sum_m x_m (R_{mi})_j \\ &\quad + \sum_{m,k} x_{mk} R_{mijk} + \sum_{l,k} x_{lk} R_{lijk} + \sum_{l,k} x_l (R_{lijk})_k.\end{aligned} \tag{2.19}$$

From $< x_i, x_{jk} >= 0$ we get (c.f. [7])

$$\begin{aligned}&\frac{1}{2}\Delta \sum_{i,j} < x_{ij}, x_{ij} > \\ &= \sum_{i,j,k} < x_{ijk}, x_{ijk} > + \sum_{i,j} < x_{ij}, (\Delta x)_{ij} > + 2K \sum_{i,j} < x_{ij}, x_{ij} > \\ &\quad + 2 \sum_{i,j,k,m} < x_{ij}, x_{mk} R_{mijk} > \\ &= \sum_{i,j,k} < x_{ijk}, x_{ijk} > + [4K(|II|^2 - 2H^2) - 4 - 2|II|^2] \\ &\quad + 2 \sum_{i,j} < x_{ij}, \vec{H}_{ij} > .\end{aligned} \tag{2.20}$$

In the following Lemma we extend a construction procedure from [1] where we applied it to eigenfunctions of the Laplacian. It follows from (2.9) that, for minimal surfaces in spheres, the position vector of M, considered as subset in $\mathbb{R}^{n+1}$, satisfies a vector valued eigenvalue equation with respect to the eigenvalue 2. For arbitrary surfaces in spheres, the following construction of the vector valued (0, 3)-tensor field W is basic for the proof of the integral inequality, stated in the introduction, and for the proof of the classification results in Theorem 3, Theorem 3.2, and Theorem 5.2.

Lemma 2.1. *We construct a totally symmetric, trace-free vector valued tensor field W by*

$$\begin{aligned} W_{ijk} := x_{ijk} + \frac{1+K}{2}\delta_{ij}x_k + \frac{1-K}{2}(\delta_{ik}x_j + \delta_{jk}x_i) \\ -\frac{1}{2}(\vec{H}_k\delta_{ij} + \vec{H}_i\delta_{jk} + \vec{H}_j\delta_{ik}), \end{aligned} \tag{2.21}$$

we have

$$\begin{aligned} \sum < W_{ijk}, W_{ijk} > = \sum < x_{ijk}, x_{ijk} > -2(K^2 - 2K + 3) \\ + 4(K-3)H^2 - 3|\mathrm{grad}\vec{H}|^2. \end{aligned} \tag{2.22}$$

Proof. The total symmetry of W can be verified using Ricci identities; the rest of the proof is straightforward.

Lemma 2.2. *Let $x : M \to S^n(1)$ be a compact surface, then we have*

$$2\int_m \sum < x_{ij}, \vec{H}_{ij} > dv = -4\int_M |\mathrm{grad}\vec{H}|^2 dv + 4\int_M H^2(K-2)dv. \tag{2.23}$$

Proof. We have the following calculation

$$\begin{aligned} & 2\int_m \sum < x_{ij}, \vec{H}_{ij} > dv \\ &= -2\int_M \sum < x_{ijj}, \vec{H}_i > dv \\ &= -2\int_M \sum < (\Delta x)_i + Kx_i, \vec{H}_i > dv \\ &= -2\int_M \sum < -2x_i + 2\vec{H}_i + Kx_i, \vec{H}_i > dv \\ &= -4\int_M |\mathrm{grad}\vec{H}|^2 dv - 2\int_M \sum < (K-2)x_i, \vec{H}_i > dv \\ &= -4\int_M |\mathrm{grad}\vec{H}|^2 dv + 2\int_M \sum < [((K-2)x_i)_i, \vec{H} > dv \\ &= -4\int_M |\mathrm{grad}\vec{H}|^2 dv + 2\int_M \sum < (K-2)\Delta x, \vec{H} > dv \\ &= -4\int_M |\mathrm{grad}\vec{H}|^2 dv + 4\int_M H^2(K-2)dv. \end{aligned}$$

3. Integral formulas

Integrating (2.20), by use of (2.22) and (2.23) we get

Proposition 3.1. *Let M be a compact surface in an n-dimensional unit sphere $S^n(1)$. Then we have the following integral formula:*

$$0 = \int_M \sum < W_{ijk}, W_{ijk} > dv + \int_M [2(3K-1)(1-K)]dv + \int_M [4H^2(2K-1)]dv - \int_M |\mathrm{grad}\vec{H}|^2 dv. \tag{3.1}$$

We apply the foregoing integral formula to prove Theorem 3.2 which, for dimension 2, extends Theorem 5.9 from [1] where we considered submanifolds with parallel mean curvature vector. The application of Theorem 3.2 in Corollary 3.4 gives new classification results in case that H^2 is constant.

Theorem 3.2. *Let M be a compact surface in an n-dimensional unit sphere $S^n(1)$. Assume that the Gauss curvature K and the mean curvature satisfy*

$$\frac{1}{2} \leq K \leq 1, \tag{3.2}$$

and

$$|\mathrm{grad}\vec{H}|^2 \leq 4H^2(2K-1), \tag{3.3}$$

then $K = 1$ and, with an appropriate choice of the local frame in the normal bundle and with e_3 being the oriented unit vector of $\vec{H}$, the components of the second fundamental form of M are given by

$$(h^3_{ij}) = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad (h^4_{ij}) = \begin{pmatrix} 0 & \sqrt{1/2}H \\ \sqrt{1/2}H & 0 \end{pmatrix},$$

$$(h^5_{ij}) = \begin{pmatrix} \sqrt{1/2}H & 0 \\ 0 & -\sqrt{1/2}H \end{pmatrix},$$

and

$$h^\beta_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta \geq 6.$$

From our foregoing results it is obvious that Theorem 3.2 is of interest only in case of non-minimal surfaces. Then condition (3.3) gives a pointwise gradient control of the extrinsic mean curvature in terms of the intrinsic Gauss curvature. From the integral formula in Proposition 3.1 and the assumptions in Theorem 3.2 it immediately follows that, to prove the Theorem, we have to discuss the case that $K = 1$, $W \equiv 0$ and

$$-|\mathrm{grad}\vec{H}|^2 + 4H^2(2K-1) = 0. \tag{3.4}$$

To discuss this case, we use the following lemma.

Lemma 3.3. *We have the following formulas*

$$x_{ijk} = -\delta_{ij}x_k - \sum_{\alpha,l} h^\alpha_{ij}h^\alpha_{kl}x_l + \sum_\alpha h^\alpha_{ij,k}e_\alpha, \tag{3.5}$$

and

$$\vec{H}_i = \sum_\alpha H^\alpha_{,i}e_\alpha - \sum_{\alpha,m} H^\alpha h^\alpha_{im}x_m, \tag{3.6}$$

where $h^\alpha_{ij,k}$ and $H^\alpha_{,i}$ are defined by

$$\sum_k h^\alpha_{ij,k}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj}\omega_{ki} + \sum_k h^\alpha_{ik}\omega_{kj} + \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}, \tag{3.7}$$

$$\sum_i H^\alpha_{,i}\omega_i = dH^\alpha + \sum_\beta H^\beta\omega_{\beta\alpha}. \tag{3.8}$$

Proof of Lemma 3.3. From $d\vec{H} = \sum \vec{H}_i\omega_i$, (3.8) and

$$\begin{aligned} d\vec{H} &= d(\sum_\alpha H^\alpha e_\alpha) = \sum_\alpha dH^\alpha e_\alpha + \sum_\alpha H^\alpha de_\alpha \\ &= \sum_{\alpha,i} H^\alpha_{,i}\omega_i e_\alpha - \sum_{\alpha,i,m} H^\alpha h^\alpha_{im}\omega_i e_m \end{aligned}$$

we get (3.6). Putting (2.8) into (2.17), we get (3.5).

Proof of Theorem 3.2. When $K = 1$, because of (3.5) and (3.6), the vector valued equation

$$0 = W_{ijk} = x_{ijk} + \delta_{ij}x_k - \frac{1}{2}(\vec{H}_k\delta_{ij} + \vec{H}_i\delta_{jk} + \vec{H}_j\delta_{ik})$$

is equivalent to the following equations (3.9) and (3.10)

$$h^\alpha_{ij,k} = \frac{1}{2}(H^\alpha_{,k}\delta_{ij} + H^\alpha_{,i}\delta_{jk} + H^\alpha_{,j}\delta_{ik}), \tag{3.9}$$

and

$$\sum_\alpha h^\alpha_{ij}h^\alpha_{mk} = \frac{1}{2}\sum_\alpha H^\alpha(h^\alpha_{km}\delta_{ij} + h^\alpha_{im}\delta_{jk} + h^\alpha_{jm}\delta_{ik}). \tag{3.10}$$

Contraction of (3.10) gives

$$\sum_\alpha H^\alpha h^\alpha_{ij} = |H|^2\delta_{ij}, \tag{3.11}$$

thus we conclude that M is pseudo-umbilical. Choosing $e_3||\vec{H}$, we have

$$h^3_{ij} = H\delta_{ij}, \quad H^3 = H, \quad H^\beta = 0, \quad \beta \geq 4. \tag{3.12}$$

By use of (3.12), we have from (3.10)

$$\sum_\alpha (h_{11}^\alpha)^2 = \sum_\alpha (h_{22}^\alpha)^2 = \frac{3}{2}H^2, \quad \sum_\alpha (h_{12}^\alpha)^2 = \frac{1}{2}H^2, \tag{3.13}$$

$$\sum_\alpha h_{11}^\alpha h_{12}^\alpha = \sum_\alpha h_{22}^\alpha h_{12}^\alpha = 0, \quad \sum_\alpha h_{11}^\alpha h_{22}^\alpha = \frac{1}{2}H^2. \tag{3.14}$$

Thus we can additionally choose

$$e_4 || \sum_\alpha h_{12}^\alpha e_\alpha, \qquad e_5 || \sum_\alpha (h_{11}^\alpha - h_{22}^\alpha) e_\alpha. \tag{3.15}$$

It is easy to check that then the components of the second fundamental form satisfy

$$(h_{ij}^3) = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} 0 & \sqrt{1/2}H \\ \sqrt{1/2}H & 0 \end{pmatrix}, \tag{3.16}$$

and

$$(h_{ij}^5) = \begin{pmatrix} \sqrt{1/2}H & 0 \\ 0 & -\sqrt{1/2}H \end{pmatrix}, \quad (h_{ij}^\beta) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta \geq 6. \tag{3.17}$$

This completes the proof of Theorem 3.2.

Corollary 3.4. *Let M be a compact surface with full codimension in an n-dimensional unit sphere $S^n(1)$. Assume that the function H^2 is constant and that the Gauss curvature K and the mean curvature satisfy*

$$\frac{1}{2} \leq K \leq 1, \tag{3.2}$$

and

$$|\mathrm{grad}\vec{H}|^2 \leq 4H^2(2K-1). \tag{3.3}$$

Then $K = 1$ and

(i) either $n = 3$, $H = 0$ and M is totally geodesic;
(ii) or $n = 5$, $H = \sqrt{2}$ and M is a Veronese surface in $S^4(\frac{1}{\sqrt{3}})$ in $S^5(1)$.

Proof of Corollary 3.4. We only need to prove the statements (i) and (ii).

By use of (3.16), we have

$$\begin{aligned}\frac{1}{2}\Delta\Big(\sum_{i,j}(h_{ij}^3)^2\Big) &= \sum_{i,j,k}(h_{ij,k}^3)^2 + \sum_{i,j,k} h_{ij}^3 h_{ij,kk}^3 \\ &= \sum_{i,j,k}(h_{ij,k}^3)^2 + \sum_{i,k} H h_{ii,kk}^3.\end{aligned} \tag{3.18}$$

If $H \neq 0$ (but not necessarily constant), we have the following formula (see Yau [18])

$$\sum_i h_{ii,kl}^3 = 2H_{kl} - \frac{1}{2H}\sum_{\alpha\neq 3}\Big(\sum_i h_{ii,k}^\alpha\Big)\Big(\sum_i h_{ii,l}^\alpha\Big), \tag{3.19}$$

which implies

$$H\sum_{i,k} h^3_{ii,kk} = 2H\Delta H - \frac{1}{2}\sum_{k,\alpha\neq 3}(\sum_i h^\alpha_{ii,k})(\sum_i h^\alpha_{ii,k}). \tag{3.20}$$

We insert (3.20) into (3.18); by use of (3.9) and (3.16) we get

$$\begin{aligned} 0 &= \int_M \sum\sum_{i,j,k}(h^3_{ij,k})^2 dv + 2\int_M H\Delta H dv - \int_M 2\sum_{\alpha\neq 3,i}(H^\alpha_{,i})^2 dv \\ &= \int_M [|\mathrm{grad} H|^2 - 2\sum_{\alpha\neq 3,i}(H^\alpha_{,i})^2] dv. \end{aligned} \tag{3.21}$$

(i) When $H =$ constant, from (3.21) we get

$$H^\alpha_{,i} = 0, \quad 1 \le i \le 2, \quad 3 \le \alpha \le n. \tag{3.22}$$

Thus $\vec{H}$ is parallel in the normal bundle (see Yau [18]), and we have $H = 0$ or $H = \sqrt{2}$ from (3.4), (3.6) and (3.11).

If $K = 1$ and $H = 0$, we know that $n = 3$ and M is totally geodesic; if $K = 1$ and $H = \sqrt{2}$, M is a Veronese surface in $S^4(\frac{1}{\sqrt{3}})$ in $S^5(1)$ from Yau's classification result of surfaces with parallel mean curvature vector (see [18]). Thus (ii) follows from our foregoing results.

4. Integral inequalities

We are going to prove the Integral Inequalities in the Introduction; Theorem 2 is an immediate consequence. To state Lemma 4.1 below, we define

$$\tilde{h}^\alpha_{ij} = h^\alpha_{ij} - H^\alpha \delta_{ij}, \tag{4.1}$$

$$\tilde{\sigma}_{\alpha\beta} = \sum_{i,j}\tilde{h}^\alpha_{ij}\tilde{h}^\beta_{ij}, \tag{4.2}$$

$$\rho^2 = \sum_{\alpha,i,j}(\tilde{h}^\alpha_{ij})^2 = |II|^2 - 2H^2. \tag{4.3}$$

Lemma 4.1. *Using the foregoing notation, we have:*

(i)

$$|\mathrm{grad}\vec{H}|^2 = \sum_{\alpha,i}(H^\alpha_{,i})^2 + \sum_{\alpha,\beta}H^\alpha H^\beta\tilde{\sigma}_{\alpha\beta} + 2H^4. \tag{4.4.a}$$

(ii)

$$H^2\rho^2 = (\sum_\alpha (H^\alpha)^2)(\sum_\beta \tilde{\sigma}_{\beta\beta}) \ge \sum_{\alpha,\beta}H^\alpha H^\beta\tilde{\sigma}_{\alpha\beta}. \tag{4.4.b}$$

Proof. (i) This follows from (3.6) and (4.2).

Now we can prove the integral inequalities in the Introduction. We use (2.6), (4.3) and Lemma 4.1 to reformulate Proposition 3.1:

$$\begin{aligned} 0 &= \int_M \sum < W_{ijk}, W_{ijk} > dv + \int_M [\rho^2(2 - \frac{3}{2}\rho^2)]dv \\ &\quad + \int_M [2H^2(\rho^2 + 2H^2) - |\mathrm{grad}\vec{H}|^2]dv \\ &= \int_M \sum < W_{ijk}, W_{ijk} > dv + \int_M [H^2\rho^2 - \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta}]dv \\ &\quad + \int_M [\rho^2(2 - \frac{3}{2}\rho^2)dv + \int_M [H^2\rho^2 - \sum_{\alpha,i}(H^\alpha_{,i})^2]dv \\ &\geq + \int_M [\rho^2(2 - \frac{3}{2}\rho^2)]dv + \int_M [H^2\rho^2 - \sum_{\alpha,i}(H^\alpha_{,i})^2]dv. \end{aligned} \tag{4.5}$$

This gives one version of the integral inequality in the Introduction; the other one follows from (2.6) and the definition of ρ^2.

5. Special classes of surfaces

In this section we will consider two special classes of surfaces which satisfy the inequality

$$\int_M [H^2\rho^2 - \sum_{\alpha,i}(H^\alpha_{,i})^2]dv \geq 0.$$

The left hand term appears in the last line in (4.5). First, we recall the definition of Willmore surfaces and apply (4.5) to this class. Secondly, we consider surfaces with parallel mean curvature vector.

Definition (see [17] or [9]). *Let M be a surface in $S^n(1)$, M is called a Willmore surface if it satisfies*

$$\Delta^\perp H^\alpha + \sum_{\beta,i,j} h^\alpha_{ij} h^\beta_{ij} H^\beta - 2H^2 H^\alpha = 0, \quad 3 \leq \alpha \leq n, \tag{5.1}$$

where $\Delta^\perp$ is the Laplacian in the normal bundle of M.

The following lemma can be found in [9]; in Remark 1.1 on p. 204 of this paper one can find several references for examples of Willmore surfaces which are non-minimal.

Lemma 5.1 (see Lemma 2.6 of [9]). *Let $x : M \to S^n(1)$ be a compact Willmore surface in an n-dimensional unit sphere $S^n(1)$. Then*

$$\int_M \sum_{\alpha,i}(H^\alpha_{,i})^2 dv = \int_M \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} dv. \tag{5.2}$$

Proof of Theorem 3. We prove version (ii). Lemma 4.1, Theorem 4.2 and Lemma 5.1 give

$$
\begin{aligned}
0 &= \int_M \sum\sum_{i,j,k} < W_{ijk}, W_{ijk} > dv + \int_M \rho^2(2 - \frac{3}{2}\rho^2)dv \\
&\quad + \int_M 2[H^2\rho^2 - \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta}]dv \\
&\geq \int_M \rho^2(2 - \frac{3}{2}\rho^2)dv. \qquad (5.3)
\end{aligned}
$$

In particular, if

$$
0 \leq \rho^2 \leq \frac{4}{3}, \qquad (5.4)
$$

then either $\rho^2 = 0$ and M is totally umbilic, or $\rho^2 = \frac{4}{3}$. In the latter case, we have from (5.2)

$$
W \equiv 0, \quad H^2\rho^2 = \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta}. \qquad (5.5)
$$

By use of (3.5), (3.6) and (2.21), we directly check that the tangential part W^T of W is given by

$$
\begin{aligned}
0 = W^T_{ijk} &= -\delta_{ij}x_k - \sum h^\alpha_{ij}h^\alpha_{kl}x_l + \frac{1+K}{2}\delta_{ij}x_k + \frac{1-K}{2}(\delta_{ik}x_j + \delta_{jk}x_i) \\
&\quad + \frac{1}{2}\sum_\alpha(h^\alpha_{km}\delta_{ij} + h^\alpha_{im}\delta_{jk} + h^\alpha_{jm}\delta_{ik})H^\alpha x_m. \qquad (5.6)
\end{aligned}
$$

(5.6) implies

$$
\begin{aligned}
\sum_\alpha h^\alpha_{ij}h^\alpha_{kl} &= -\delta_{ij}\delta_{kl} + \frac{1+K}{2}\delta_{ij}\delta_{kl} + \frac{1-K}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) \\
&\quad + \frac{1}{2}\alpha(h^\alpha_{kl}\delta_{ij} + h^\alpha_{il}\delta_{jk} + h^\alpha_{jl}\delta_{ik})H^\alpha. \qquad (5.7)
\end{aligned}
$$

Contraction of (5.7) gives

$$
\sum_\alpha H^\alpha h^\alpha_{ij} = H^2\delta_{ij}. \qquad (5.8)
$$

From (5.8), we get

$$
\sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} = \sum_{\alpha,\beta} H^\alpha H^\beta(h^\alpha_{ij}h^\beta_{ij} - 2H^\alpha H^\beta) = 0. \qquad (5.9)
$$

(5.5) and (5.9) imply $H = 0$, thus $n = 4$, and M is the Veronese surface given by Example 1. This proves Theorem 3. Theorem 2 can be proved similarly.

In an analogous way we get a result for surfaces with parallel mean curvature vector in spheres. It follows from (3.6) that this is the case if and only if $H^\alpha_{,i} = 0$ for all α. Then (4.5) gives the following integral inequality.

$$0 \geq \int_M [\rho^2(2 + H^2 - \frac{3}{2}\rho^2)]dv = \int_M 2[1 - (K - H^2)][3K - (1 + 2H^2)]dv.$$

As a consequence we get Theorem 5.2 where the interval for K, compared with the intervals $(1.5.b)$, $(1.5.c)$ prescribed in the case of Willmore surfaces, can be enlarged.

Theorem 5.2. *Let M be a compact surface with parallel mean curvature vector in an n-dimensional unit sphere $S^n(1)$. Assume that the curvature functions K, H^2 and ρ^2 satisfy one of the following two (equivalent) inequalities:*

(i)

$$\frac{1}{3}(1 + 2H^2) \leq K \leq 1 + H^2;$$

(i)

$$0 \leq \rho^2 \leq \frac{4}{3} + \frac{2}{3}H^2.$$

Then either $n = 3$, $\rho^2 = 0$, $K = 1 + H^2$ and M is totally umbilical in $S^3(1)$, or $n = 4$, $\rho^2 = \frac{4}{3}$, $K = \frac{1}{3}$ and M is the Veronese surface given by Example 1.

The proof follows the lines of that of Theorem 3.

References

[1] Benko, K., Kothe, M., Semmler, K.-D., Simon, U.: Eigenvalues of the Laplacian and curvature. Colloq. Math. **42**, 19–31 (1979)
[2] Bolton, J., Woodward, L.M.: On the Simon conjecture for minimal immersions with S^1-symmetry. Math. Z. **200**, 111–121 (1988)
[3] Calabi, E.: Minimal immersions of surfaces in euclidean spheres. J. Diff. Geom. **1**, 111–125 (1967)
[4] Chern, S.S., Do Carmo, M., Kobayashi, S.: Minimal submanifolds of a sphere with second fundamental form of constant length. In: Functional Analysis and Related Fields (F.Brower, ed.), pp. 59–75, Springer-Verlag, Berlin, 1970
[5] Dillen, F.: Minimal immersions of surfaces in spheres. Arch. Math. **49**, 94–96 (1987)
[6] Itoh, T.: A characterization of the generalized Veronese surfaces. Proc. Amer. Math. Soc. **104**, 571–576 (1988)
[7] Kozlowski, M., Simon, U.: Minimal immersions of 2-manifolds into spheres. Math. Z. **186**, 377–382 (1984)
[8] Lawson, H.B.: Local rigidity theorem for minimal hypersurfaces. Ann. Math. (2) **89**, 187–197 (1969)
[9] Li, H.: Willmore surfaces in S^n. Ann. Global Anal. Geom. **21**, 203–213 (2002)
[10] Liu, H.L., Simon, U., Verstraelen, L., Wang, C.P.: The third fundamental form metric for hypersurfaces in nonflat space forms. J. Geom. **65**, 130–142 (1999)
[11] Okayasu, T.: Minimal immersions of curvature pinched 2-manifolds into spheres. Kodai Math. J. **10**, 116–126 (1987)

[12] Pavlista, T.: Geometrische Abschätzungen kleiner Eigenwerte des Laplaceoperators. Dissertation FB Mathematik TU Berlin, 1984

[13] Sakamoto, K.: On the curvature of minimal 2-spheres in spheres. Math. Z. **228**, 605–627 (1998)

[14] Simon, U.: Eigenvalues of the Laplacian and minimal immersions into spheres, In: Differential Geometry (L.A. Cordero, ed.), Pitman, Montreal, 1985 pp. 115–120

[15] Simon, U., Vrancken, L., Wang, C.P., Wiehe, M.: Intrinsic and extrinsic geometry of ovaloids and rigidity. In: Geometry and Topology of Submanifolds X (eds.: W. Chen et al.), World Scientific, Singapore, 2000 pp. 284–293

[16] Simons, J.: Minimal varieties in riemannian manifolds. Ann. of Math. **88**, 62–105 (1968)

[17] Weiner, J.: On a problem of Chen, Willmore, et al. Indiana Univ. Math. J. **27**, 19–35 (1978)

[18] Yau, S.-T.: Submanifolds with constant mean curvature I. Amer. J. Math. **96**, 346–366 (1974)