The second variational formula for Willmore submanifolds in S^n *

Dedicated to Professor S. S. Chern at the occasion of his 90th birthday

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Abstract: In [17] the third author presented Moebius geometry for submanifolds in S^n and calculated the first variational formula of the Willmore functional by using Moebius invariants. In this paper we present the second variational formula for Willmore submanifolds. As an application of these variational formulas we give the standard examples of Willmore hypersurfaces $\{W_k^m := S^k(\sqrt{(m-k)/m}) \times S^{m-k}(\sqrt{k/m}),\ 1 \le k \le m-1\}$ in S^{m+1} (which can be obtained by exchanging radii in the Clifford tori $S^k(\sqrt{k/m}) \times S^{m-k}(\sqrt{(m-k)/m})$) and show that they are stable Willmore hypersurfaces. In case of surfaces in S^3, the stability of the Clifford torus $S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}})$ was proved by J. L. Weiner in [18]. We give also some examples of m-dimensional Willmore submanifolds in an n-dimensional unit sphere S^n.

Keywords: Willmore functional, second variational formula, stability, Willmore conjecture
2000 MR Subject Classification 53A30, 53B25

§0. Introduction

Let M be an m-dimensional submanifold immersed in S^n. The Willmore functional W is defined by

$$W(M) = \left(\frac{m}{m-1}\right)^{\frac{m}{2}} \int_M \left(S - m\|H\|^2\right)^{\frac{m}{2}} dM, \tag{0.1}$$

*The second author is partially supported by the Alexander von Humboldt Research Fellowship and the US-China Cooperative Research NSF Grant INT-9906856 and the third author is partially supported by 973-project, DFG466-CHV-II3/127/0, Qiushi Award and RFDP.

which is a conformal invariant under Moebius (or conformal) transformations of S^n (see [3], [17], [19]), where S is the square of the length of the second fundamental form, H is the mean curvature vector and dM is the volume element of the induced metric on M. In case of a surface in S^3, this functional is equivalent to the Willmore functional $W_e(M) := 4\int_M H^2 dM$ for surfaces in R^3. The so-called Willmore conjecture states that $W_e(M) \geq 8\pi^2$ for any embedded torus in R^3, well-studied (cf. [2], [9], [19], [20], [21]) since 1965. An equivalent version of the Willmore conjecture states that $W(T^2) \geq 8\pi^2$ for any embedded torus T^2 in S^3 and the equality holds if and only if T^2 is Moebius equivalent to the Clifford torus. Many efforts and partial results have been obtained for this conjecture, but up to now it is still open.

From the analytic point of view, the second variational formula of the Willmore functional might be very important to solve the Willmore conjecture. In 1978, J. L. Weiner gave the second variational formula for minimal surfaces, which is a special class of Willmore surfaces (cf. [18]). But it is rather complicated to give the second variation formula for general Willmore surfaces by using Euclidean invariants. In [12] M. Peterson derives such a formula. In [10] and [11] B. Palmer calculated the second variation of a Willmore surface in S^3 by using its conformal Gauss map and used the formula to study the nonexistence of stable Willmore surfaces under some conditions.

Since the Willmore functional (0.1) is invariant under the Moebius group (cf. [3], [17]), one can use the framework of Moebius geometry and Moebius invariants to calculate the second variational formula. It is the key point of this paper. For any submanifold M in S^n we can introduce a Moebius invariant metric g on M. Then the Willmore functional is exactly the volume functional of g. The third author computed the first variation and got the Euler - Lagrange equations in [17]. Submanifolds in S^n satisfying these equations are called Willmore submanifolds or Moebius minimal submanifolds. In this paper we give the second variational formula of the Willmore functional for submanifolds in S^n by using Moebius invariants. Although this formula looks very complicated, in case of surfaces in S^3 (which is the most important case) the formula is in good form (cf. §2, (2.43)). Using the Euler-Lagrange equations we find the standard examples of Willmore hypersurfaces $\{W_k^m := S^k(\sqrt{(m-k)/m}) \times S^{m-k}(\sqrt{k/m}),\ 1 \leq k \leq m-1\}$ in S^{m+1}, which is (euclidean) minimal if and only if $2k = m$. In some sense, W_k^m can be considered as the dual hypersurface of the standard minimal hypersurface $S^k(\sqrt{k/m}) \times S^{m-k}(\sqrt{(m-k)/m})$ in S^{m+1}. We show that the hypersurface W_k^m are stable Willmore hypersurfaces.

We organize this paper as follows. In §1 we give Moebius invariants and local formulas in Moebius geometry for submanifolds in S^n. In §2 we calculate the second variation formula for Willmore submanifolds in S^n. As an application we prove in §3 that $\{W_k^m\}$ are stable Willmore hypersurfaces.

§1. Moebius invariants and local formulas for submanifolds in S^n

Let $x_0 : M \to S^n$ be an m-dimensional compact submanifold with boundary ∂M, $\{e_1, ..., e_m\}$ be a local orthonormal basis of TM with respect to the induced metric $dx_0 \cdot dx_0$ and $\{\theta_1, ..., \theta_m\}$ be its dual basis. Let $\{e_{m+1}, ..., e_n\}$ be the local normal orthonormal vector field. We make use of the following convention on the ranges of indices:

$$1 \leq i, j, k, \cdots \leq m; \quad m+1 \leq \alpha, \beta, \gamma, \ldots \leq n$$

and we shall agree that repeated indices are summed over the respective ranges. Then the structure equation of x_0 can be written as

$$\begin{aligned} dx_0 &= \sum \theta_i e_i \\ de_i &= \sum \theta_{ij} e_j + \sum h^\alpha_{ij}\theta_j e_\alpha - \theta_i x_0 \\ de_\alpha &= -\sum h^\alpha_{ij}\theta_j e_i + \sum \theta_{\alpha\beta} e_\beta. \end{aligned}$$

The quantities $I = dx_0 \cdot dx_0, II = \sum h^\alpha_{ij}\theta_i \otimes \theta_j e_\alpha$ and $H = \frac{1}{m}\sum h^\alpha_{ii} e_\alpha$ are the first, the second fundamental form and the mean curvature vector of x_0 in S^n, respectively. We define the function $\rho : M \to R$ by

$$\rho = \sqrt{\frac{m}{m-1}}\|II - HI\|. \tag{1.1}$$

The metric $g = \rho^2 dx_0 \cdot dx_0$ is called a Moebius metric which is invariant under Moebius transformations in S^n (cf.[17]); it is positive definite at any non-umbilical point. Then the Willmore functional in (0.1) is exactly the Moebius volume functional for g:

$$W(M) := \int_M \rho^m dM = Vol_g(M), \tag{1.2}$$

where dM is the volume element for the metric $dx_0 \cdot dx_0$. In this paper, it is our purpose to calculate the second variation in the framework of Moebius geometry. We need the following notation and local formulas. For more details we refer to [17].

Let R^{n+2}_1 be the Lorentz space with the inner product $<,>$ given by

$$< X, W >= -x^0 w^0 + x^1 w^1 + \cdots + x^{n+1} w^{n+1},$$

where $X = (x^0, x^1, \cdots, x^{n+1}), W = (w^0, w^1, \cdots, w^{n+1}) \in R^{n+2}$. The half cone in R^{n+2}_1 is defined as

$$C^{n+1}_+ = \{X \in R^{n+2}_1 | < X, X >= 0, x^0 > 0\}.$$

For the immersion $x_0 : M \to S^n$ we define

$$Y = \rho(1, x_0) : M \to C^{n+1}_+. \tag{1.3}$$

If $\tilde{x}_0 : M \to S^n$ is Moebius equivalent to x_0, then we have $\tilde{Y} = YT$ for some $T \in O(n+1, 1)$. Thus

$$g =< dY, dY >= \rho^2 dx_0 \cdot dx_0 \tag{1.4}$$

is a Moebius invariant. In the following we assume that x_0 is an immersion without umbilical point, which implies that g is positive definite on M. Let $E_i = \rho^{-1} e_i$, then $\{E_i\}$ is an orthonormal basis with respect to the metric g, and its dual basis is $\{\omega_i = \rho\theta_i\}$. Set

$$Y_i := E_i(Y),\ N := -\frac{1}{m}\Delta Y - \frac{1}{2m^2} < \Delta Y, \Delta Y > Y,$$

$$E_\alpha := (H^\alpha, e_\alpha + H^\alpha x_0), \tag{1.5}$$

where Δ is the Laplacian of the metric g and $H = H^\alpha e_\alpha$ is the mean curvature vector with respect to the induced metric $dx_0 \cdot dx_0$.

Lemma 1.1([17]) $\{Y, N, Y_i, E_\alpha\}$ *satisfy the following conditions*

$$< Y, Y >=< N, N >= 0, < Y, N >= 1,\ < Y_i, Y_j >= \delta_{ij};$$

$$< Y, Y_i >=< N, Y_i >=< Y, E_\alpha >=< N, E_\alpha >= 0;$$

$$< E_\alpha, Y_i >= 0,\ < E_\alpha, E_\beta >= \delta_{\alpha\beta}.$$

Lemma 1.1 shows that $\{Y, N, Y_i, E_\alpha\}$ forms a Moebius moving frame in R_1^{n+2} along M. The structure equations can be written as

$$\begin{array}{l} dY = \sum \omega_i Y_i \\ dN = \sum \psi_i Y_i + \sum \phi_\alpha E_\alpha \\ dY_i = -\psi_i Y - \omega_i N + \sum \omega_{ij} Y_j + \sum \omega_{i\alpha} E_\alpha \\ dE_\alpha = -\phi_\alpha Y - \sum \omega_{i\alpha} Y_i + \sum \omega_{\alpha\beta} E_\beta. \end{array} \tag{1.6}$$

By differentiating these equations and using Cartan's lemma, we obtain

$$\psi_i = \sum A_{ij}\omega_j,\ A_{ij} = A_{ji};\ \omega_{i\alpha} = \sum B^\alpha_{ij}\omega_j,\ B^\alpha_{ij} = B^\alpha_{ji};\ \phi_\alpha = \sum C^\alpha_i \omega_i.$$

We have the following equations:

$$\sum_i B^\alpha_{ii} = 0,\ \sum_{\alpha,i,j} \left(B^\alpha_{ij}\right)^2 = \frac{m-1}{m},\ \sum_j B^\alpha_{ij,j} = -(m-1)C^\alpha_i. \tag{1.7}$$

The relations between these Moebius invariants and Euclidean invariants are given by

$$A_{ij} = -\rho^{-2}(Hess_{ij}(log\rho) - e_i(log\rho)e_j(log\rho) - H^\alpha h^\alpha_{ij})$$

$$-\frac{1}{2}\rho^{-2}(|\nabla log\rho|^2 - 1 + \sum_\alpha (H^\alpha)^2)\delta_{ij}, \tag{1.8}$$

$$B^\alpha_{ij} = \rho^{-1}(h^\alpha_{ij} - H^\alpha\delta_{ij}), \tag{1.9}$$

$$C^\alpha_i = -\frac{1}{m-1}B^\alpha_{ij,j} = -\rho^{-2}(H^\alpha_{,i} + (h^\alpha_{ij} - H^\alpha\delta_{ij})e_j(log\rho)). \tag{1.10}$$

§2. The second variational formula of the Willmore functional

In this section we calculate the second variation of the Willmore functional (or the Moebius volume functional) defined by (1.2) or (0.1). Since the variation of the volume depends only on the normal component of the variational vector field (cf. [17]), we will consider the normal variation.

Let $x : M \times R \to S^n$ be a smooth variation of x_0 such that $x(\cdot, t) = x_0$ and $dx_t(TM) = dx_0(TM)$ on ∂M for each (small) t. These two boundary conditions disappear if $\partial M = \emptyset$. For each t we denote by$\{e_i\}$ a local orthonormal basis for TM with respect to $dx_t \cdot dx_t$, by $\{\theta_i\}$ its dual basis and by $\{e_\alpha\}$ a local orthonormal basis for the normal bundle of x_t. Let $Y = \rho(1, x) : M \times R \to C^{n+1}_+$ be the canonical lift of x_t and $g_t =< dY, dY >$ be the

Moebius metric of x_t. Let $\{E_i := \rho^{-1}e_i\}$ be a local orthonormal basis for g_t with dual basis $\{\omega_i = \rho\theta_i\}$. Then the volume for g_t can be written as

$$W(t) := Vol_{g_t}(M) = \int_M \omega_1 \wedge \cdots \wedge \omega_m. \tag{2.1}$$

From Lemma 1.1 in section 1, we can choose a moving frame

$$\{Y, N, Y_1, \cdots, Y_m, E_{m+1}, \cdots, E_n\}$$

in R_1^{n+2} along $M \times R$, which satisfies the conditions in Lemma 1.1 for each t. Let d denote the differential operator on $M \times R$, then we can find 1-forms

$$\{V, V_\alpha, \Psi_i, \Phi_\alpha, \Omega_i, \Omega_{ij}, \Omega_{i\alpha}, \Omega_{\alpha\beta}\}$$

on $M \times R$ with $\Omega_{ij} = -\Omega_{ji}$ and $\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$ such that

$$dY = VY + \Omega_i Y_i + V_\alpha E_\alpha, \tag{2.2}$$

$$dN = -VN + \Psi_i Y_i + \Phi_\alpha E_\alpha, \tag{2.3}$$

$$dY_i = -\Psi_i Y - \Omega_i N + \Omega_{ij} Y_j + \Omega_{i\alpha} E_\alpha, \tag{2.4}$$

$$dE_\alpha = -\Phi_\alpha - V_\alpha N - \Omega_{i\alpha} Y_i + \Omega_{\alpha\beta} E_\beta. \tag{2.5}$$

Taking the differentials of these equations we get

$$dV = \Psi_i \wedge \Omega_i + \Phi_\alpha \wedge V_\alpha, \tag{2.6}$$

$$d\Omega_i = \Omega_{ij} \wedge \Omega_j + V \wedge \Omega_i - V_\alpha \wedge \Omega_{i\alpha}, \tag{2.7}$$

$$dV_\alpha = \Omega_{\alpha\beta} \wedge V_\beta + \Omega_i \wedge \Omega_{i\alpha} + V \wedge V_\alpha, \tag{2.8}$$

$$d\Psi_i = \Omega_{ij} \wedge \Psi_j - \Phi_\alpha \wedge \Omega_{i\alpha} + \Psi_i \wedge V, \tag{2.9}$$

$$d\Phi_\alpha = \Omega_{\alpha\beta} \wedge \Phi_\beta + \Psi_i \wedge \Omega_{i\alpha} + \Phi_\alpha \wedge V, \tag{2.10}$$

$$d\Omega_{ij} = \Omega_{ik} \wedge \Omega_{kj} + \Omega_{i\alpha} \wedge \Omega_{\alpha j} - \Psi_i \wedge \Omega_j - \Omega_i \wedge \Psi_j, \tag{2.11}$$

$$d\Omega_{i\alpha} = \Omega_{ij} \wedge \Omega_{j\alpha} + \Omega_{i\beta} \wedge \Omega_{\beta\alpha} - \Psi_i \wedge V_\alpha - \Omega_i \wedge \Phi_\alpha, \tag{2.12}$$

$$d\Omega_{\alpha\beta} = \Omega_{\alpha\gamma} \wedge \Omega_{\gamma\beta} + \Omega_{\alpha i} \wedge_{i\beta} - \Phi_\alpha \wedge V_\beta - V_\alpha \wedge \Phi_\beta. \tag{2.13}$$

Since $Y = \rho(1, x)$, if we write the normal variation vector field of x in TS^n by

$$\frac{\partial x}{\partial t} = \rho^{-1} v_\alpha e_\alpha, \tag{2.14}$$

then by (1.5) we can find a function $v : M \times R \to R$ such that

$$\frac{\partial Y}{\partial t} = vY + v_\alpha E_\alpha. \tag{2.15}$$

From (2.2), (2.15) and the fact that $d = \omega_i E_i + dt\frac{\partial}{\partial t}$ on $C^\infty(M \times R)$, we get

$$V = vdt, \;\; V_\alpha = v_\alpha dt, \;\; \Omega_i = \omega_i. \tag{2.16}$$

Since $T^*(M\times R)=T^*M\oplus T^*R$ we can write

$$\Psi_i=\psi_i+a_idt, \Phi_\alpha=\phi_\alpha+b_\alpha dt, \tag{2.17}$$

$$\Omega_{ij}=\omega_{ij}+P_{ij}dt,\ \Omega_{i\alpha}=\omega_{i\alpha}+L_{i\alpha}dt,\ \Omega_{\alpha\beta}=\omega_{\alpha\beta}+Q_{\alpha\beta}dt, \tag{2.18}$$

where $\{a_i,b_\alpha,P_{ij},Q_{\alpha\beta}\}$ are local functions with $P_{ij}=-P_{ji}$ and $Q_{\alpha\beta}=-Q_{\beta\alpha}$. Let $\{B^\alpha_{ij},A_{ij},C^\alpha_i\}$ be Moebius invariants for x_t defined in §1. If we denote by d_M the exterior differential operator on T^*M, then we have $d=d_M+dt\wedge\frac{\partial}{\partial t}$ on $T^*(M\times R)$. It follows from (2.6), (2.16) and (2.17) (comparing the terms in $T^*(M)\wedge dt$) that

$$a_i=-v_{,i}+v_\alpha C^\alpha_i, \tag{2.19}$$

where $v_{,i}:=E_i(v)$. Similarly we get from (2.8), (2.16), (2.17), (2.18) that

$$L_{i\alpha}=v_{\alpha,i} \tag{2.20}$$

and get from (2.7), (2.16), (2.18) that

$$\frac{\partial\omega_i}{\partial t}=(P_{ij}+v\delta_{ij}-v_\alpha B^\alpha_{ij})\omega_j. \tag{2.21}$$

By a direct calculation in a similar way, we obtain from (2.9)∼(2.13) that

$$\frac{\partial\omega_{ij}}{\partial t}=(P_{ij,k}+B^\alpha_{ik}v_{\alpha,j}-B^\alpha_{jk}v_{\alpha,i}-a_i\delta_{kj}+a_j\delta_{ik})\omega_k, \tag{2.22}$$

$$\frac{\partial\psi_i}{\partial t}=(a_{i,j}+P_{ik}A_{kj}+v_{\alpha,i}C^\alpha_j-b_\alpha B^\alpha_{ij}-vA_{ij})\omega_j \tag{2.23}$$

$$\frac{\partial\phi_\alpha}{\partial t}=(b_{\alpha,i}+Q_{\alpha\beta}C^\beta_i-A_{ij}v_{\alpha,j}+a_jB^\alpha_{ji}-vC^\alpha_i)\omega_i, \tag{2.24}$$

$$\frac{\partial\omega_{i\alpha}}{\partial t}=(v_{\alpha,ij}+P_{ik}B^\alpha_{kj}-B^\beta_{ij}Q_{\beta\alpha}+A_{ij}v_\alpha+b_\alpha\delta_{ij})\omega_j \tag{2.25}$$

$$\frac{\partial\omega_{\alpha\beta}}{\partial t}=(Q_{\alpha\beta,i}+v_{\beta,j}B^\alpha_{ji}-v_{\alpha,j}B^\beta_{ji}+v_\beta C^\alpha_i-v_\alpha C^\beta_i)\omega_i, \tag{2.26}$$

where $\{v_{\alpha,i}\}$ are covariant derivatives of $\{v_\alpha\}$. Since $\phi_\alpha=C^\alpha_i\omega_i$ and $\psi_i=A_{ij}\omega_j$, from (2.21), (2.23) and (2.24) we have

$$\frac{\partial C^\alpha_i}{\partial t}=b_{\alpha,i}+Q_{\alpha\beta}C^\beta_i+a_jB^\alpha_{ij}-A_{ij}v_{\alpha,j}+P_{ij}C^\alpha_j+B^\beta_{ij}C^\alpha_jv_\beta-2vC^\alpha_i, \tag{2.27}$$

$$\frac{\partial A_{ij}}{\partial t}=a_{i,j}+P_{ik}A_{kj}-P_{kj}A_{ki}+v_{\alpha,i}C^\alpha_j-B^\alpha_{ij}b_\alpha+A_{ik}B^\alpha_{kj}v_\alpha-2vA_{ij}. \tag{2.28}$$

Since $\omega_{i\alpha}=B^\alpha_{ij}\omega_j$, from (2.21) and (2.25) we have

$$\frac{\partial B^\alpha_{ij}}{\partial t}=v_{\alpha,ij}-vB^\alpha_{ij}+P_{ik}B^\alpha_{kj}-P_{kj}B^\alpha_{ki}-B^\beta_{ij}Q_{\beta\alpha}+v_\beta B^\alpha_{ik}B^\beta_{kj}+A_{ij}v_\alpha+b_\alpha\delta_{ij}. \tag{2.29}$$

Multiplying B^α_{ij} to (2.29) and using (1.7) we get

$$\frac{m-1}{m}v = B^\alpha_{ij}v_{\alpha,ij} + B^\alpha_{ik}B^\beta_{kj}B^\alpha_{ji}v_\beta + A_{ij}B^\alpha_{ij}v_\alpha. \tag{2.30}$$

Making use of (2.21), (2.30), (1.7), Green's formula and the conditions on ∂M, we get by noting (2.34) of [17]

$$W'(t) = \frac{m^2}{m-1}\int_M \left\{-(m-1)C^\alpha_{i,i} + B^\beta_{ik}B^\alpha_{kj}B^\beta_{ji} + A_{ij}B^\alpha_{ij}\right\} v_\alpha dM_t, \tag{2.31}$$

where dM_t is volume element of g_t.

We summarize as follows

Proposition 2.1. **([17])** *$x_0 : M \to S^n$ is a Willmore submanifold (i.e., its Willmore functional is stationary) if and only if*

$$-(m-1)C^\alpha_{i,i} + B^\beta_{ik}B^\alpha_{kj}B^\beta_{ji} + A_{ij}B^\alpha_{ij} = 0. \tag{2.32}$$

Since $\frac{\partial}{\partial t}\circ d_M - d_M \circ \frac{\partial}{\partial t} = 0$ (or equivalently $d^2 = (d_M + dt\wedge\frac{\partial}{\partial t})^2 = 0$) and

$$C^\alpha_{i,j}\omega_j = d_M C^\alpha_i + C^\alpha_j\omega_{ji} + C^\beta_i\omega_{\beta\alpha},$$

we get

$$\frac{\partial C^\alpha_{i,j}}{\partial t} = -C^\alpha_{i,k}\frac{\partial\omega_k}{\partial t}(E_j) + (\frac{\partial C^\alpha_i}{\partial t})_{,j} + C^\alpha_k\frac{\partial\omega_{ki}}{\partial t}(E_j) + C^\beta_i\frac{\partial\omega_{\beta\alpha}}{\partial t}(E_j). \tag{2.33}$$

By a direct calculation and use of (2.21), (2.22), (2.26) and (2.33), we get

$$\begin{aligned}\frac{\partial C^\alpha_{i,i}}{\partial t}v_\alpha &= -(C^\alpha_i P_{ki})_{,k}v_\alpha - vC^\alpha_{i,i}v_\alpha + v_\beta B^\beta_{ki}C^\alpha_{i,k}v_\alpha + \left(\frac{\partial C^\alpha_i}{\partial t}\right)_{,i} v_\alpha \\ &+C^\alpha_k B^\beta_{ki}v_{\beta,i}v_\alpha + a_k C^\alpha_k v_\alpha - m a_k C^\alpha_k v_\alpha - C^\beta_i Q_{\alpha\beta,i}v_\alpha - B^\alpha_{ki}C^\beta_i v_{\beta,k}v_\alpha \\ &+B^\beta_{ki}C^\beta_i v_{\alpha,k}v_\alpha - C^\beta_i C^\alpha_i v_\beta v_\alpha + \sum_{i,\beta}\left(C^\beta_i\right)^2\sum_\alpha v_\alpha^2.\end{aligned} \tag{2.34}$$

By a straightforward calculation and using (2.27), (2.19), we get from (2.34) that

$$\begin{aligned}&\frac{\partial C^\alpha_{i,i}}{\partial t}v_\alpha = -(C^\alpha_i P_{ki}v_\alpha)_{,k} + \left(\frac{\partial C^\alpha_i}{\partial t}v_\alpha\right)_{,i} + (vB^\alpha_{ki}v_{\alpha,i})_{,k} \\ &+(m-1)(vC^\alpha_k v_\alpha)_{,k} - (C^\beta_i Q_{\alpha\beta}v_\alpha)_{,i} - (b_\alpha v_{\alpha,i})_{,i} - mvC^\alpha_{i,i}v_\alpha \\ &+C^\beta_{i,i}Q_{\alpha\beta}v_\alpha + C^\alpha_{i,k}B^\beta_{ki}v_\alpha v_\beta + b_\alpha v_{\alpha,ii} - vB^\alpha_{ki}v_{\alpha,ik} + A_{ik}v_{\alpha,k}v_{\alpha,i} \\ &-2C^\alpha_k B^\beta_{ki}v_\beta v_{\alpha,i} + 2vC^\alpha_i v_{\alpha,i} - m\sum_k\left(\sum_\alpha C^\alpha_k v_\alpha\right)^2 \\ &+C^\beta_i B^\beta_{ik}v_{\alpha,k}v_\alpha + \sum_{i,\beta}(C^\beta_i)^2\sum_\alpha v_\alpha^2.\end{aligned} \tag{2.35}$$

From the conditions that $x(.,t) = x_0$ and $dx_t(TM) = dx_0(TM)$ on ∂M for each t, we see that $v_\alpha|_{\partial M} = 0$ and

$$0 = \frac{\partial}{\partial t}(dx_t) = d(\frac{\partial x}{\partial t}) = \rho^{-1} dv_\alpha e_\alpha \tag{2.36}$$

at the boundary ∂M. It follows from (2.35) and Green's formula that

$$\begin{aligned}
\Gamma_1 := & \int_M \{\frac{\partial}{\partial t}|_{t=0}(C^\alpha_{i,i})\} v_\alpha dM \\
= & \int_M \{-mvC^\alpha_{i,i}v_\alpha + C^\beta_{i,i}Q_{\alpha\beta}v_\alpha + C^\alpha_{i,k}B^\beta_{ki}v_\alpha v_\beta + b_\alpha v_{\alpha,ii} \\
& -vB^\alpha_{ki}v_{\alpha,ik} + A_{ik}v_{\alpha,k}v_{\alpha,i} - 2C^\alpha_k B^\beta_{ki}v_\beta v_{\alpha,i} + 2vC^\alpha_i v_{\alpha,i} \\
& -m\sum_k \left(\sum_\alpha C^\alpha_k v_\alpha\right)^2 + C^\beta_i B^\beta_{ik}v_{\alpha,k}v_\alpha + \sum_{i,\beta}(C^\beta_i)^2 \sum_\alpha v^2_\alpha\} dM.
\end{aligned} \tag{2.37}$$

Using (2.28), (2.29), (2.19) and the facts that $P_{ij} = -P_{ji}$ and $\sum_j B^\alpha_{ij,j} = -(m-1)C^\alpha_i$, we get

$$\begin{aligned}
& \frac{\partial}{\partial t}\left(B^\beta_{ik}B^\alpha_{kj}B^\beta_{ji} + A_{ij}B^\alpha_{ij}\right) v_\alpha \\
& = (vB^\alpha_{ij}v_{\alpha,j})_{,i} - (m-1)(vC^\alpha_i v_\alpha)_{,i} + (a_i B^\alpha_{ij}v_\alpha)_{,j} \\
& -2(m-1)vC^\alpha_{i,i}v_\alpha + 2B^\beta_{ij}B^\alpha_{jk}v_{\beta,ik}v_\alpha + (B^\beta_{ik}B^\beta_{kj} + A_{ij})v_{\alpha,ij}v_\alpha - vB^\alpha_{ij}v_{\alpha,ij} \\
& +(m-1)C^\beta_{i,i}Q_{\alpha\beta}v_\alpha + 2(m-1)vC^\alpha_j v_{\alpha,j} + C^\gamma_i B^\alpha_{ij}v_{\gamma,j}v_\alpha - C^\gamma_i B^\alpha_{ij}v_{\alpha,j}v_\gamma \\
& +\left(3B^\beta_{ij}B^\alpha_{jk}B^\gamma_{kl}B^\beta_{li} + 4B^\alpha_{ik}B^\gamma_{kj}A_{ji} + (m-1)C^\alpha_i C^\gamma_i\right) v_\alpha v_\gamma \\
& +\sum_{i,j}\sum_\beta (B^\beta_{ij}b_\beta)(\sum_\alpha B^\alpha_{ij}v_\alpha) + (\sum_\alpha b_\alpha v_\alpha)\left(\sum_{\beta,i,j}(B^\beta_{ij})^2 + tr(A)\right) \\
& +\sum_{i,j} A^2_{ij}\sum_\alpha v^2_\alpha + B^\beta_{ik}A_{kj}B^\beta_{ji}(\sum_\alpha v^2_\alpha).
\end{aligned} \tag{2.38}$$

Using the boundary conditions and Green's theorem we have

$$\begin{aligned}
\Gamma_2 := & \int_M \{\frac{\partial}{\partial t}|_{t=0}(B^\beta_{ik}B^\alpha_{kl}B^\beta_{lj} + A_{ij}B^\alpha_{ij})\} v_\alpha dM \\
= & \int_M \{-2(m-1)vC^\alpha_{i,i}v_\alpha + 2B^\beta_{ij}B^\alpha_{jk}v_{\beta,ik}v_\alpha + (B^\beta_{ik}B^\beta_{kj} + A_{ij})v_{\alpha,ij}v_\alpha \\
& -vB^\alpha_{ij}v_{\alpha,ij} + (m-1)C^\beta_{i,i}Q_{\alpha\beta}v_\alpha + 2(m-1)vC^\alpha_j v_{\alpha,j} + C^\gamma_i B^\alpha_{ij}v_{\gamma,j}v_\alpha \\
& -C^\gamma_i B^\alpha_{ij}v_{\alpha,j}v_\gamma + \left(3B^\beta_{ij}B^\alpha_{jk}B^\gamma_{kl}B^\beta_{li} + 4B^\alpha_{ik}B^\gamma_{kj}A_{ji} + (m-1)C^\alpha_i C^\gamma_i\right) v_\alpha v_\gamma \\
& +\sum_{ij}\sum_\beta (B^\beta_{ij}b_\beta)(\sum_\alpha B^\alpha_{ij}v_\alpha) + (\sum_\alpha b_\alpha v_\alpha)(\frac{m-1}{m} + tr(A)) \\
& +\sum_{ij} A^2_{ij}\sum_\alpha v^2_\alpha + B^\beta_{ik}A_{kj}B^\beta_{ji}(\sum_\alpha v^2_\alpha)\} dM,
\end{aligned} \tag{2.39}$$

where

$$b_\alpha = -\frac{1}{m}\left(\Delta v_\alpha + B^\alpha_{ik}B^\beta_{ki}v_\beta + \frac{1}{2m}(1 + m^2\kappa)v_\alpha\right), \tag{2.40}$$

$$tr(A) = \frac{1}{2m} + \frac{m}{2}\kappa, \tag{2.41}$$

where κ is the normalized scalar curvature of the metric $g = \rho^2 dx_0 \cdot dx_0$. The function v is given by (2.15) and (2.30). Thus for a Willmore submanifold x_0, we get from (2.31) and (1.7) that

$$\begin{aligned} W''(0) &= -m^2 \int_M \frac{\partial C^{\alpha}_{i,i}}{\partial t}|_{t=0} v_\alpha dM \\ &+ \frac{m^2}{m-1}\int_M \left(\frac{\partial}{\partial t}|_{t=0}(B^{\beta}_{ik}B^{\alpha}_{kj}B^{\beta}_{ji} + A_{ij}B^{\alpha}_{ij})\right) v_\alpha dM. \end{aligned}$$

Thus the formula of the second variation is given by (2.37) and (2.39). We conclude that

Theorem 2.1. *Let $x : M \to S^n$ be a compact Willmore submanifold without boundary. Then the second variation formula of the Moebius volume functional is given by*

$$\begin{aligned} W''(0) = \int_M \Bigg\{ & m^2(m-2)vC^{\alpha}_{i,i}v_\alpha - m^2 C^{\alpha}_{i,j}B^{\beta}_{ji}v_\alpha v_\beta + \frac{m^2(m-2)}{m-1}vB^{\alpha}_{ij}v_{\alpha,ij} \\ & -m^2 A_{ij}v_{\alpha,i}v_{\alpha,j} + \frac{m^2(2m-1)}{m-1}C^{\beta}_i B^{\alpha}_{ij}v_{\beta,j}v_\alpha + m^2(m+1)\sum_i\left(\sum_\alpha C^{\alpha}_i v_\alpha\right)^2 \\ & -m^2 C^{\beta}_i B^{\beta}_{ij}v_{\alpha,j}v_\alpha - m^2\sum_{i,\beta}\left(C^{\beta}_i\right)^2\sum_\alpha v_\alpha^2 + \frac{2m^2}{m-1}B^{\beta}_{ij}B^{\alpha}_{jk}v_{\beta,ik}v_\alpha \\ & + \frac{m^2}{m-1}(B^{\beta}_{ik}B^{\beta}_{kj} + A_{ij})v_{\alpha,ij}v_\alpha - \frac{m^2}{m-1}C^{\beta}_i B^{\alpha}_{ij}v_{\alpha,j}v_\beta \\ & + \frac{m^2}{m-1}\left(3B^{\gamma}_{ij}B^{\alpha}_{jk}B^{\beta}_{kl}B^{\gamma}_{li} + 4B^{\alpha}_{ik}B^{\beta}_{kj}A_{ji}\right)v_\alpha v_\beta + \frac{m^2}{m-1}\sum_{ij}A^2_{ij}\sum_\alpha v^2_\alpha \\ & + \frac{m^2}{m-1}B^{\beta}_{ik}A_{kj}B^{\beta}_{ji}\left(\sum_\alpha v^2_\alpha\right) + m\sum_\alpha(\Delta v_\alpha)^2 + \frac{m(m-2)}{m-1}B^{\alpha}_{ij}B^{\beta}_{ij}\Delta v_\alpha v_\beta \\ & + \left(\frac{m(m-2)}{m-1}tr(A) - 1\right)v_\alpha\Delta v_\alpha - \frac{m}{m-1}\sum_\alpha\left(\sum_{ij\beta}B^{\alpha}_{ij}B^{\beta}_{ij}v_\beta\right)^2 \\ & - \left(1 + \frac{2m}{m-1}tr(A)\right)\sum_{ij}\left(\sum_\alpha B^{\alpha}_{ij}v_\alpha\right)^2 - tr(A)\left(1 + \frac{m}{m-1}tr(A)\right)\sum_\alpha v^2_\alpha\Bigg\} dM. \end{aligned} \tag{2.42}$$

In case of the surface in S^3, we omit all α and β because the codimension now is one, the formula is reduced to

Corollary 2.1. *For a Willmore surface $x_0 : M \to S^3$, the second variational formula is* *given by*

$$W''(0) = \int_M \{2(\Delta f)^2 + 2f\Delta f + 12\sum_{ij} C_i B_{ij} f_j f + 4\sum_{ij} A_{ij} f_{ij} f$$
$$-4\sum_{ij} A_{ij} f_i f_j + (4\sum_{ij} A_{ij}^2 + 4\sum_i C_i^2 + \frac{7}{8} + K - 2K^2)\} dM, \tag{2.43}$$

where K is Gaussian curvature of the Moebius metric $g = \rho^2 dx_0 \cdot dx_0$.

Remark 2.1. *The second variational formula for Willmore surfaces in S^3 might be important to solve the Willmore conjecture. As far as we know the only known stable example of a Willmore torus is the Clifford torus. From the existence result of L. Simon in [15], we know that the Willmore conjecture is true if one can show that the only stable Willmore torus embedded in S^3 is the Clifford torus.*

§3. Willmore tori in S^{m+1} and their stabilities

In this section we present a class of important examples of Willmore hypersurfaces called Willmore tori. As an application of Theorem 2.1 we show that they are stable Willmore hypersurfaces.

Let R^{m+2} be the (m+2)-dimensional Euclidean space with inner product $<,>$. We write $R^{m+2} = R^{k+1} \times R^{m-k+1}, 1 \le k \le m-1$. For any vector $\xi \in R^{m+2}$ there is a unique decomposition $\xi = \xi_1 + \xi_2$ with $\xi_1 \in R^{k+1}$ and $\xi_2 \in R^{m-k+1}$. For another vector $\eta = \eta_1 + \eta_2$ the inner product of them can be written as $<\xi, \eta> = <\xi_1, \eta_1> + <\xi_2, \eta_2>$. Let $\xi_1 : S^k \to R^{k+1}$ and $\xi_2 : S^{m-k} \to R^{m-k+1}$ be standard embeddings of unit spheres. Let $x : S^k(a_1) \times S^{m-k}(a_2) \to S^{m+1} \subset R^{m+2}$ be the embedded hypersurface $x = a_1\xi_1 + a_2\xi_2$ with $a_1^2 + a_2^2 = 1$. It is easy to check that

(i) the unit normal vector of $M := S^k(a_1) \times S^{m-k}(a_2)$ in S^{m+1} is given by

$$e_{m+1} = -a_2\xi_1 + a_1\xi_2;$$

(ii) the second fundamental form of M is given by

$$II = -<dx, de_{m+1}> = a_1a_2(<d\xi_1, d\xi_1> - <d\xi_2, d\xi_2>);$$

(iii) the induced metric of M is given by

$$I = a_1^2|d\xi_1|^2 + a_2^2|d\xi_2|^2.$$

If we take $\{e_i\}$ and $\{\omega_i\}$such that

$$d(a_1\xi_1) = \sum_{i=1}^{k} \omega_i e_i, \ d(a_2\xi_2) = \sum_{j=k+1}^{n} \omega_j e_j,$$

then we have

$$I=\sum_{i=1}^{m}\omega_i^2,\quad II=\sum_{i=1}^{k}\frac{a_2}{a_1}\omega_i^2-\sum_{j=k+1}^{m}\frac{a_1}{a_2}\omega_j^2:=h_{ij}\omega_i\omega_j, \tag{3.1}$$

where

$$h_{ij}=\begin{cases}\frac{a_2}{a_1}\delta_{ij}, & if\ 1\le i,j\le k,\\ \\ -\frac{a_1}{a_2}\delta_{ij}, & if\ k+1\le i,j\le n.\end{cases} \tag{3.2}$$

Theorem 3.1. *Let $W=S^k(a_1)\times S^{m-k}(a_2)$ be the hypersurface imbedded into S^{m+1}, where $a_1^2+a_2^2=1$. Then W is a Willmore hypersurface if and if*

$$a_1=\sqrt{\frac{m-k}{m}},\quad a_2=\sqrt{\frac{k}{m}}.$$

Proof. From (3.1) we see that

$$S:=\sum_{i,j}^{m}h_{ij}^2=k\left(\frac{a_2}{a_1}\right)^2+(m-k)\left(\frac{a_1}{a_2}\right)^2, \tag{3.3}$$

$$H:=\frac{1}{m}\sum_{i=1}^{m}h_{ii}=\frac{1}{m}\left(k\frac{a_2}{a_1}-(m-k)\frac{a_1}{a_2}\right), \tag{3.4}$$

$$\rho^2=\frac{m}{m-1}(S-mH^2). \tag{3.5}$$

Substituting (3.2) and (3.5) into (1.8)-(1.10) we know that the Willmore condition

$$-(m-1)C_{i,i}+B_{ik}B_{kj}B_{ji}+A_{ij}B_{ij}=0$$

is equivalent to the equation

$$(m-k)\left(\frac{a_2}{a_1}\right)^6+(2m-3k)\left(\frac{a_2}{a_1}\right)^4+(m-3k)\left(\frac{a_2}{a_1}\right)^2-k=0. \tag{3.6}$$

From the equations (3.6) and $a_1^2+a_2^2=1$ we get $a_1=\sqrt{\frac{m-k}{m}}$ and $a_2=\sqrt{\frac{k}{m}}$.

We call

$$W_k^m=S^k\left(\sqrt{\frac{m-k}{m}}\right)\times S^{m-k}\left(\sqrt{\frac{k}{m}}\right),\ 1\le k\le m-1,$$

a *Willmore torus.*

Remark 3.1. *It is remarkable that the Willmore torus W_k^m is (Euclidean) minimal if and only if $2k=m$. We note that W_k^m can be obtained by exchanging the radii $\sqrt{k/m}$ and $\sqrt{m-k/m}$ in the Clifford minimal torus $S^k\left(\sqrt{\frac{k}{m}}\right)\times S^{m-k}\left(\sqrt{\frac{m-k}{m}}\right)$.*

Remark 3.2. *It is known that any minimal surface in S^n is a Willmore surface (see Weiner [18]) (also see Pinkall's examples of non-minimal Willmore surfaces [13]). Theorem 3.1 shows that m-dimensional ($m\ge 3$) minimal hypersurfaces are not Willmore hypersurfaces in general.*

Remark 3.3. *In [6], the second author proved an integral inequality of Simons' type for m-dimensional compact Willmore hypersurfaces in S^{m+1} and gave a characterization of Willmore tori W_k^m by use of his integral inequality.*

From now on we study the stability of Willmore tori W_k^m defined in Theorem 3.1. For W_k^m we get from (3.2), (3.3), (3.4) and (3.5) that

$$h_{ij} = \begin{cases} \sqrt{\frac{k}{m-k}}\delta_{ij}, & if\ 1 \le i, j \le k, \\ -\sqrt{\frac{m-k}{k}}\delta_{ij}, & if\ k+1 \le i, j \le m, \end{cases} \tag{3.7}$$

$$S = \frac{k^3 + (m-k)^3}{k(m-k)},\ H = -\frac{m-2k}{\sqrt{k(m-k)}},\ \rho^2 = \frac{m^2}{m-1}. \tag{3.8}$$

From (1.8) and (1.9) we have

$$A_{ij} = \rho^{-2} H h_{ij} + \frac{1}{2}\rho^{-2}(1 - H^2)\delta_{ij}$$

$$= \begin{cases} \rho^{-2}\frac{3km-k^2-m^2}{2k(m-k)}\delta_{ij}, & if\ 1 \le i, j \le k, \\ \rho^{-2}\frac{m^2-k^2-km}{2k(m-k)}\delta_{ij}, & if\ k+1 \le i, j \le m, \end{cases} \tag{3.9}$$

$$tr(A) = \frac{(m-1)(m^2 - 3km + 3k^2)}{2km(m-k)}, \tag{3.10}$$

$$B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij}) = \begin{cases} \rho^{-1}\sqrt{\frac{m-k}{k}}\delta_{ij}, & if\ 1 \le i, j \le k \\ -\rho^{-1}\sqrt{\frac{k}{m-k}}\delta_{ij}, & if\ k+1 \le i, j \le m, \end{cases} \tag{3.11}$$

and $\sum B_{ij}^2 = \frac{m-1}{m}$. From the last equation in (1.7) we obtain $C_i = 0$. We are going to calculate

$$W''(0) = -m^2\Gamma_1 + \frac{m^2}{m-1}\Gamma_2,$$

where $\Gamma_1 = \int_M \frac{\partial C_{i,i}}{\partial t}|_{t=0} f dM$, $\Gamma_2 = \int_M \{\frac{\partial}{\partial t}|_{t=0}(B_{ik}B_{kl}B_{li} + A_{ij}B_{ji})\} f dM$ and $f \in C^\infty(M)$ is the normal component of the variational vector field. From (2.40) we have

$$b = -\frac{1}{m}\left(\Delta f + \frac{(m-1)(m^2+k^2-km)}{2km(m-k)} f\right). \tag{3.12}$$

Substituting (3.9), (3.11) and (3.12) into (2.37), we get

$$\begin{aligned} -m^2\Gamma_1 = & \int_M \left(m(\Delta f)^2 + \frac{(m-1)(m^2+k^2-km)}{2k(m-k)} f\Delta f\right) dM \\ & - \int_M \left(\frac{(m-1)(3km-k^2-m^2)}{2k(m-k)}|\nabla_1 f|^2 + \frac{(m-1)(m^2-km-k^2)}{2k(m-k)}|\nabla_2 f|^2\right) dM \\ & + m^2 \int_M v B_{ij} f_{ij} dM. \end{aligned} \tag{3.13}$$

Substituting (3.9), (3.11) and (3.12) into (2.39), by a straightforward calculation we get

$$\begin{aligned}
\frac{m^2}{m-1}\Gamma_2 &= -\frac{m^2-k^2+km}{2k(m-k)}\int_M f\Delta f dM + \frac{5m^2+5k^2-9km}{2k(m-k)}\int_M f\Delta_1 f dM \\
&+\frac{5k^2+m^2-km}{2k(m-k)}\int_M f\Delta_2 f dM - \frac{m^2}{m-1}\int_M vB_{ij}f_{ij}dM \\
&+\Big(2km(m-k)(m^2+7km-7k^2) - m(m^2-k^2+km)(k^2+m^2-km) \\
&+ k(3km-m^2-k^2)^2 + (m-k)(m^2-km-k^2)^2\Big)\frac{m-1}{4k^2m^2(m-k)^2}\int_M f^2 dM.
\end{aligned} \tag{3.14}$$

In (3.13) and (3.14) we denote by Δ the Laplace operator on W_k^m with respect to the Moebius metric $g = \rho^2 dx\cdot dx$. We write $g = g_1 \oplus g_2$ according to the decomposition $M_k^m := M_1 \times M_2 := S^k\left(\sqrt{\frac{m-k}{m}}\right)\times S^{m-k}\left(\sqrt{\frac{k}{m}}\right)$. We denote by $\{\Delta_1,\Delta_2\}$ the Laplace operators of $\{g_1, g_2\}$ and by $\{\nabla_1,\ \nabla_2\}$ the gradient operators on M_1 and M_2, respectively. From (2.30), (3.9) and (3.11), we have

$$vB_{ij}f_{ij} = \frac{1}{m}\left(\sqrt{\frac{m-k}{k}}\Delta_1 f - \sqrt{\frac{k}{m-k}}\Delta_2 f\right)^2. \tag{3.15}$$

From(3.13), (3.14) and (3.15) we get

$$\begin{aligned}
W''(0) &= \int_M \Bigg(m(\Delta f)^2 + \frac{m(k^2+m^2-km-2m)}{2k(m-k)}f\Delta f \\
&+\frac{m(m-2)}{m-1}\left(\sqrt{\frac{m-k}{k}}\Delta_1 f - \sqrt{\frac{k}{m-k}}\Delta_2 f\right)^2 \\
&+\frac{m(3km-k^2-m^2)+6(m-k)^2}{2k(m-k)}f\Delta_1 f \\
&+\frac{m(m^2-km-k^2)+6k^2}{2k(m-k)}f\Delta_2 f + \frac{2(m-1)}{m}f^2\Bigg)dM.
\end{aligned} \tag{3.16}$$

We denote the Laplace operators with respect to the Euclidean metric $dx\cdot dx$ by Δ_M, Δ_{M_1} and Δ_{M_2} on W_k^m, M_1 and M_2, respectively. Since $\rho = constant$ and the Moebius metric $g = \rho^2 dx\cdot dx$, we have $\Delta_M = \rho^2\Delta$, $\Delta_{M_1} = \rho^2\Delta_1$, $\Delta_{M_2} = \rho^2\Delta_2$ and $\Delta_M = \Delta_{M_1} + \Delta_{M_2}$. From

(3.8) and (3.16), we have

$$
\begin{aligned}
W''(0) = \quad & \frac{m-1}{2k(m-k)m^3}\int_M \Big(2k(m-k)(m-1)(\Delta_M f)^2 \\
& +m^2(k^2+m^2-km-2m)f\Delta_M f \\
& +2(m-2)\left((m-k)\Delta_{M_1}f - k\Delta_{M_2}f\right)^2 \\
& +m\Big(m(3km-k^2-m^2)+6(m-k)^2\Big)f\Delta_{M_1}f \\
& +m\Big(m(m^2-km-k^2)+6k^2\Big)f\Delta_{M_2}f \\
& +4k(m-k)m^2f^2\big)\,dM.
\end{aligned}
\tag{3.17}
$$

Let λ_i, λ'_i and μ_{ij} be the eigenvalues of Laplace operators $\Delta_{M_1}, \Delta_{M_2}$ and Δ_M, respectively, then we have

$$
\lambda_i = i(k+i-1)\frac{n}{n-k},\ \lambda'_j = j(n-k+j-1)\frac{n}{k},\ \mu_{ij} = \lambda_i + \lambda'_j,
$$

where i and j are nonnegative integers. Let f_i, f'_i be eigenfunctions corresponding to λ_i and λ'_i, respectively, then $g_{ij}(p,q) = f_i(p)f'_j(q)$ $((p,q)\in M_1\times M_2)$ is an eigenfunction corresponding to μ_{ij}. For any $f\in C^\infty(M_k^m)$ we have the decomposition of f into eigenfuctions

$$
f = \sum_{i+j\neq 0} c_{ij}g_{ij} + c_0,
\tag{3.18}
$$

where c_{ij} and c_0 are constants. Thus we have

$$
\Delta_{M_1}f = -\sum_{i+j\neq 0}\lambda_i c_{ij}g_{ij},\ \Delta_{M_2}f = -\sum_{i+j\neq 0}\lambda'_j c_{ij}g_{ij},\ \Delta_M f = -\sum_{i+j\neq 0}\mu_{ij}c_{ij}g_{ij}.
\tag{3.19}
$$

Substituting (3.18) and (3.19) into formula (3.17), we get

$$
\begin{aligned}
W''(0) \quad & \geq \tfrac{m-1}{2k(m-k)m^3}\int_M \textstyle\sum_{i+j\neq 0}\Big(2k(m-k)(m-1)\mu_{ij}^2 \\
& -m^2(k^2+m^2-km-2m)\mu_{ij} + 2(m-2)\Big((m-k)\lambda_i - k\lambda'_j\Big)^2 \\
& -m\left(m(3km-k^2-m^2)+6(m-k)^2\right)\lambda_i \\
& -m\left(m(m^2-km-k^2)+6k^2\right)\lambda'_j + 4k(m-k)k^2\big)\,c_{ij}^2g_{ij}^2dM
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
& = \tfrac{m-1}{2k(m-k)m^3}\textstyle\sum_{i+j\neq 0}\int_M\{2(m-k)(m^2-2m+k)\lambda_i^2 \\
& -2(2m^3+km^3-6km^2+3k^2m)\lambda_i \\
& +2k(m^2-m-k)(\lambda'_j)^2 - 2m(m^3-m^2-km^2+3k^2)\lambda'_j \\
& +4k(m-k)\lambda_i\lambda'_j + 4k(m-k)m^2\}c_{ij}^2g_{ij}^2dM.
\end{aligned}
\tag{3.21}
$$

If we set

$$\begin{aligned} A(i,j) &= 2(m-k)(m^2-2m+k)\lambda_i^2 - 2(2m^3+km^3-6km^2+3k^2m)\lambda_i \\ &+2k(m^2-m-k)(\lambda_j')^2 - 2m(m^3-m^2-km^2+3k^2)\lambda_j' \\ &+4k(m-k)\lambda_i\lambda_j' + 4k(m-k)m^2, \end{aligned}$$

then one can easily verify that

$$\begin{aligned} &A(i,j) = \Big(\tfrac{km}{(m-k)^2}(m^2-2m+k)i(k+i-1) + (\tfrac{2km}{m-k}j(m-k+j-1) - 2km)\Big) \\ &\cdot \Big(\tfrac{m(m-k)}{k^2}(m^2-m-k)j(m-k+j-1) + (\tfrac{2(m-k)m}{k}i(k+i-1) - 2(m-k)m)\Big) \\ &-\tfrac{m^2(m^2-2m+k)(m^2-m-k)}{k(m-k)}ij(k+i-1)(m-k+j-1). \end{aligned} \tag{3.22}$$

From (3.22) it is not difficult to see that $A(i,j) \geq 0$ and $A(i,j) = 0$ if and only if $(i,j) = (1,0), (0,1)$ or $(1,1)$. Thus we have proved the main result in this section:

Theorem 3.2. *All Willmore tori $S^k\left(\sqrt{\frac{m-k}{m}}\right) \times S^{m-k}\left(\sqrt{\frac{k}{m}}\right) \to S^{m+1}, 1 \leq k \leq m-1$, are stable.*

Now we would like to propose the following generalized Willmore conjecture in S^{m+1} (cf. Pinkall [14] and Kobayashi [5]):

Generalized Willmore Conjecture: Let G_k be a m-dimensional manifold which is diffeomorphic to $S^k \times S^{m-k}$ and $x : G_k \to S^{m+1}$ be an imbedding, where $1 \leq k \leq m-1$. Set

$$\tau_k(x) = \left(\frac{m}{m-1}\right)^{\frac{m}{2}} \int_{G_k} \left(S - mH^2\right)^{\frac{m}{2}} dM, \tag{3.23}$$

where dM is the volume element of G_k with respect to the induced metric $dx \cdot dx$, S the square of the length of the second fundamental form and H the mean curvature of x. Then

$$\tau_k(x) \geq \frac{4\pi^{\frac{m+2}{2}}(m-k)^{\frac{k}{2}}k^{\frac{m-k}{2}}}{m^{\frac{m}{2}-2}(m-1)\Gamma(\frac{k+1}{2})\Gamma(\frac{m-k+1}{2})} \tag{3.24}$$

and equality holds if and only if $x(M)$ is Moebius equivalent to W_k^m. Here Γ is the gamma function and the term on the right hand side of (3.24) is the Moebius volume of W_k^m.

Remark 3.4. *When $m = 2$, the above generalized Willmore conjecture reduces to the Willmore conjecture.*

§4. Examples of Willmore Submanifolds in S^n

In this section, we will give some examples of Willmore submanifolds in a unit sphere S^n. We first prove

Theorem 4.1. *Let $x : M \to S^n$ be a m-dimensional submanifold in an n-dimensional unit sphere. Then $x(M)$ is a m-dimensional Willmore submanifold in S^n if and only if, for any α with $m+1 \le \alpha \le n$,*

$$\begin{array}{c} -\rho^{m-2}[SH^\alpha + H^\beta h^\beta_{ij} h^\alpha_{ij} - h^\alpha_{ij} h^\beta_{ik} h^\beta_{kj} - m|\vec{H}|^2 H^\alpha] \\ +(m-1)\Delta(\rho^{m-2}H^\alpha) - (\rho^{m-2})_{,ij}(mH^\alpha\delta_{ij} - h^\alpha_{ij}) = 0, \end{array} \tag{4.1}$$

where Δ and $(.)_{,ij}$ are the Laplacian and covariant derivatives with respect to the induced metric $dx \cdot dx$, respectively.

Proof. From [17], we have the following relations between the connections of the Moebius metric $\rho^2 dx \cdot dx$ and the induced metric $dx \cdot dx$, respectively,

$$\omega_{ij} = \theta_{ij} + (\ln\rho)_i\theta_j - (\ln\rho)_j\theta_i, \qquad \omega_{\alpha\beta} = \theta_{\alpha\beta}. \tag{4.2}$$

By use of (1.10), we get

$$\begin{array}{rcl} \rho C^\alpha_{i,j}\theta_j &=& C^\alpha_{i,j}\omega_j \\ &=& dC^\alpha_i + C^\alpha_j\omega_{ji} + C^\beta_i\omega_{\beta\alpha} \\ &=& dC^\alpha_i + C^\alpha_j\theta_{ji} + C^\beta_i\theta_{\beta\alpha} + [(\ln\rho)_k\theta_i - (\ln\rho)_i\theta_k]\cdot C^\alpha_k, \end{array}$$

thus

$$\begin{array}{rcl} \rho C^\alpha_{i,j} &=& 2\rho^{-3}\rho_j[H^\alpha_{,i} + (h^\alpha_{ik} - H^\alpha\delta_{ik})(\ln\rho)_k] - \rho^{-2}[H^\alpha_{,ij} + (h^\alpha_{ik,j} - H^\alpha_{,j}\delta_{ik})(\ln\rho)_k] \\ && -\rho^{-2}(h^\alpha_{ik} - H^\alpha\delta_{ik})(\ln\rho)_{kj} + C^\alpha_k(\ln\rho)_k\delta_{ij} - (\ln\rho)_i C^\alpha_j. \end{array} \tag{4.3}$$

Letting $i = j$ and making summation over i in (4.3), we have

$$\begin{array}{rcl} \sum_i C^\alpha_{i,i} &=& -\rho^{-3}\Delta H^\alpha - \rho^{-3}(h^\alpha_{ik} - H^\alpha\delta_{ik})(\ln\rho)_{ki} \\ && -2(m-2)\rho^{-3}H^\alpha_{,i}(\ln\rho)_i - (m-3)\rho^{-3}(\ln\rho)_i(\ln\rho)_j(h^\alpha_{ij} - H^\alpha\delta_{ij}). \end{array} \tag{4.4}$$

On the other hand, we have from (1.8) and (1.9)

$$\begin{array}{rcl} A_{ij}B^\alpha_{ij} + B^\beta_{ik}B^\beta_{kj}B^\alpha_{ij} &=& -\rho^{-3}(\ln\rho)_{ij}(h^\alpha_{ij} - H^\alpha\delta_{ij}) + \rho^{-3}(\ln\rho)_i(\ln\rho)_j(h^\alpha_{ij} - H^\alpha\delta_{ij}) \\ && +\rho^{-3}[h^\beta_{ik}h^\beta_{kj}h^\alpha_{ij} - H^\beta h^\beta_{ij}h^\alpha_{ij} - SH^\alpha + m|\vec{H}|^2H^\alpha]. \end{array} \tag{4.5}$$

Putting (4.5) into (2.32), we get

$$\begin{array}{l} \frac{m-1}{\rho^{m+1}}\{-\frac{\rho^{m-2}}{m-1}[SH^\alpha + H^\beta h^\beta_{ij}h^\alpha_{ij} - h^\alpha_{ij}h^\beta_{ik}h^\beta_{kj} - m|\vec{H}|^2H^\alpha] \\ \quad +\rho^{m-2}\Delta H^\alpha + \frac{m-2}{m-1}\rho^{m-2}(\ln\rho)_{,ij}(H^\alpha\delta_{ij} - h^\alpha_{ij}) + 2(m-2)\rho^{m-2}(\ln\rho)_iH^\alpha_{,i} \\ \quad +\frac{(m-2)^2}{m-1}\rho^{m-2}(\ln\rho)_i(\ln\rho)_j(h^\alpha_{ij} - H^\alpha\delta_{ij})\} = 0. \end{array} \tag{4.6}$$

By a direct computation we can check the following identity

$$\begin{array}{rl} & -\frac{1}{m-1}(\rho^{m-2})_{ij}(mH^\alpha\delta_{ij} - h^\alpha_{ij}) + \Delta(\rho^{m-2}H^\alpha) \\ = & \frac{(m-2)^2}{m-1}\rho^{m-2}(\ln\rho)_i(\ln\rho)_j(h^\alpha_{ij} - H^\alpha\delta_{ij}) + \frac{m-2}{m-1}\rho^{m-2}(\ln\rho)_{ij}(h^\alpha_{ij} - H^\alpha\delta_{ij}) \\ & +2(m-2)\rho^{m-2}(\ln\rho)_iH^\alpha_{,i} + \rho^{m-2}\Delta H^\alpha. \end{array} \tag{4.7}$$

Thus (4.6) is equivalent to (4.1) by use of (4.7). We complete the proof of Theorem 4.1. The

Gauss equations of an m-dimensional submanifold $x: M \to S^n$ give

$$R_{ij} = (m-1)\delta_{ij} + mH^\beta h_{ij}^\beta - h_{ik}^\beta h_{kj}^\beta, \tag{4.8}$$

$$R = m(m-1) + m^2|\vec{H}|^2 - S, \tag{4.9}$$

where R_{ij} and R are the Ricci curvature and scalar curvature of M, respectively, with respect to the induced metric $dx \cdot dx$. By use of (4.8) and (4.9) we have,

$$SH^\alpha + H^\beta h_{ij}^\beta h_{ij}^\alpha - h_{ij}^\alpha h_{ik}^\beta h_{kj}^\beta - n|\vec{H}|^2 H^\alpha = (R_{ij} - (m-1)H^\beta h_{ij}^\beta)(h_{ij}^\alpha - H^\alpha \delta_{ij}). \tag{4.10}$$

Combining (4.1) and (4.10), we prove **Corollary 4.1.** *$x: M \to S^n$ is an m-dimensional Willmore submanifold in an n-dimensional unit sphere if and only if, for any α with $m+1 \leq \alpha \leq n$,*

$$\begin{aligned} &-\rho^{m-2}(R_{ij} - (m-1)H^\beta h_{ij}^\beta)(h_{ij}^\alpha - H^\alpha\delta_{ij}) \\ &\quad +(m-1)\Delta(\rho^{m-2}H^\alpha) - (\rho^{m-2})_{,ij}(mH^\alpha\delta_{ij} - h_{ij}^\alpha) = 0. \end{aligned} \tag{4.11}$$

Remark 4.1. *When $m = 2$, Corollary 4.1 implies that $x: M^2 \to S^n$ is a Willmore surface if and only if*

$$\Delta H^\alpha + h_{ij}^\alpha h_{ij}^\beta H^\beta - 2|\vec{H}|^2 H^\alpha = 0, \qquad 3 \leq \alpha \leq n,$$

which was first proved by Weiner in [18]. Thus all minimal surfaces in S^n are Willmore surfaces. In particular, the Veronese surfaces in S^n are Willmore surfaces. In [8], Li-Wang-Wu gave a Moebius characterization of Veronese surfaces in S^n.

Remark 4.2. *Fix the index α with $m+1 \leq \alpha \leq n$, define $\Box^\alpha : M \to R$ by*

$$\Box^\alpha f = (mH^\alpha\delta_{ij} - h_{ij}^\alpha)f_{ij},$$

where f is any smooth function on M. We know that $\Box^\alpha$ is a self-adjoint operator (cf. Cheng-Yau [4] and Li [7]). This operator naturally appears in the Willmore equations (4.1) *or* (4.11).

Theorem 4.2. *Let $x: M \to S^n$ be a m-dimensional minimal submanifold in an n-dimensional unit sphere. If M is an Einstein manifold, then $x: M \to S^n$ is a Willmore submanifold.*

Proof. From our assumption

$$H^\alpha = 0, \quad m+1 \leq \alpha \leq n, \qquad R_{ij} = \frac{R}{m}\delta_{ij}, \tag{4.12}$$

we have, by use of (4.9),

$$\rho^2 = S - m|\vec{H}|^2 = S = m(m-1) - R = \text{constant}. \tag{4.13}$$

Thus from (4.12) and (4.13) we know that (4.11) holds. We conclude that M is a Willmore submanifold.

Example 4.1. $S^m(\sqrt{\frac{2(m+1)}{m}}) \to S^{m+p}$ with $p = \frac{1}{2}(m-1)(m+2)$. Let

$$S^m(\sqrt{\frac{2(m+1)}{m}}) = \{(x_0, x_1, \cdots, x_m) \in R^{m+1}; \sum x_i^2 = \frac{2(m+1)}{m}\}.$$

Let E be the space of $(m+1) \times (m+1)$ symmetric matrices (u_{ij}), $(i, j = 1, \cdots, m)$, such that $\sum u_{ii} = 0$; it is a vector space of dimension $\frac{1}{2}m(m+3)$. We define a norm in E by $||(u_{ij})||^2 = \sum u_{ij}^2$. Let S^{m+p} with $p = \frac{1}{2}(m-1)(m+2)$ be the unit hypersphere in E. The mapping of $S^m\sqrt{\frac{2(m+1)}{m}})$ into S^{m+p}, defined by

$$u_{ij} = \frac{1}{2}\sqrt{\frac{m}{m+1}}(x_i x_j - \frac{2}{m}\delta_{ij}),$$

is an isometric minimal immersion. (Actually, this gives an imbedding of the real projective space of m-dimension into S^{m+p}). We know that $S^m(\sqrt{\frac{2(m+1)}{m}})$ is an Einstein manifold, thus from Theorem 4.2 this immersion is a Willmore submanifold.

Example 4.2. $CP^m_{2m/(m+1)} \to S^{m(m+2)-1}$. We can define a minimal immersion of the m-dimensional complex projective space $CP^m_{2m/(m+1)}$ with holomorphic sectional curvature $2m/(m+1)$ into a unit sphere $S^{m(m+2)-1}$ such that the usual coordinate functions of $R^{m(m+2)}$ are all independent hermitian harmonic functions of degree 1 on $CP^m_{2m/(m+1)}$ (see Wallach [16]). We also know that $CP^m_{2m/(m+1)}$ is an Einstein manifold with Ricci curvature m. Therefore from Theorem 4.2 we conclude that this immersion is a Willmore submanifold.

Proposition 4.1. *Let $M = S^{m_1}(a_1) \times \cdots \times S^{m_r}(a_r)$ be the submanifold imbedded into S^{m+r-1}, where $m_1 + \cdots + m_r = m$. Then M is a Willmore submanifold if and only if*

$$a_i = \sqrt{\frac{m - m_i}{m(r-1)}}, \quad i = 1, \cdots, r. \tag{4.14}$$

Proof. Consider

$$R^{m+r} = R^{m_1+1} \times \cdots \times R^{m_r+1}, \quad m = \sum_{i=1}^r m_i,$$

$$S^{m_1}(a_1) \times \cdots \times S^{m_r}(a_r) = \{(a_1 x_1, \cdots, a_r x_r) \in R^{m+r} : |x_i| = 1, i = 1, \cdots, r\},$$

where $S^{m_i}(a_i) \subset R^{m_i+1}$, $(i = 1, \cdots, r)$, $\sum_{i=1}^r a_i^2 = 1$, is a m-dimensional submanifold in S^{m+r-1}.

Writing $x = (a_1 x_1, \cdots, a_r x_r) : M = S^{m_1}(a_1) \times \cdots \times S^{m_r}(a_r) \to S^{m+r-1}$, $x_1 \cdot x_1 = \cdots = x_r \cdot x_r = 1$. Then $r-1$ orthogonal normal vector fields of M are

$$e_{n+\lambda} = (a_{\lambda 1} x_1, \cdots, a_{\lambda r} x_r), \quad \lambda = 1, \cdots, r-1$$

where $(a_{\lambda 1}, \cdots, a_{\lambda r})$ satisfies that $(r \times r)$ matrix

$$A = \begin{pmatrix} a_1 & \cdots & a_r \\ a_{11} & \cdots & a_{1r} \\ \vdots & \cdots & \vdots \\ a_{r-11} & \cdots & a_{r-1r} \end{pmatrix}$$

is an orthogonal matrix, thus

$$\sum_{\lambda=1}^{r-1} a_{\lambda i}a_{\lambda j} = \delta_{ij} - a_i a_j. \tag{4.15}$$

Therefore the first fundamental form and the second fundamental form of M are given by

$$I = dx \cdot dx = a_1^2 dx_1 \cdot dx_1 + \cdots + a_r^2 dx_r \cdot dx_r,$$

$$II = -\sum_{\lambda=1}^{r-1} [a_1 a_{\lambda 1} dx_1 \cdot dx_1 + \cdots + a_r a_{\lambda r} dx_r \cdot dx_r] e_{m+\lambda}.$$

We get the components of the second fundamental form

$$(h_{ij}^{m+\lambda}) = \begin{pmatrix} -\frac{a_{\lambda 1}}{a_1}E_1 & & \\ & \ddots & \\ & & \frac{a_{\lambda r}}{a_r}E_r \end{pmatrix}, \quad \lambda = 1, \cdots, r-1 \tag{4.16}$$

where E_i is a $(m_i \times m_i)$ unit matrix. It follows that $\rho^2 = m(S - mH^2)/(m-1)$ is constant and $\nabla^\perp e_\alpha = 0$, $\Delta H^\alpha = 0$, $\alpha = m + \lambda$.

From (4.1), we know that M is a Willmore submanifold if and only if

$$SH^\alpha + H^\beta h_{ij}^\beta h_{ij}^\alpha - h_{ij}^\alpha h_{ik}^\beta h_{kj}^\beta - m|\vec{H}|^2 H^\alpha = 0, \quad n+1 \le \alpha = m+\lambda \le n+r-1. \tag{4.17}$$

Putting

$$H^{m+\lambda} = \frac{1}{m}\sum_{i=1}^{r} \frac{a_{\lambda i}}{a_i} m_i, \quad 1 \le \lambda \le r-1, \quad S = \sum_{i=1} m_i \frac{1-a_i^2}{a_i^2},$$

$$-\sum_{\mu,i,j,k} h_{ij}^{m+\lambda} h_{ik}^{m+\mu} h_{kj}^{m+\mu} = \sum_i \frac{m_i}{a_i^3} a_{\lambda i}(1-a_i^2), \quad \sum_{i,j} h_{ij}^{m+\lambda} h_{ij}^{m+\mu} = \sum_i m_i \frac{a_{\lambda i} a_{\mu i}}{a_i^2}$$

into (4.17), by use of (4.15) and (4.16) we can easily check that (4.17) can be reduced to

$$\begin{aligned} \sum_i a_{\lambda i}\{-\frac{m_i}{m a_i}\sum_j m_j(1-\frac{1}{a_j^2}) - \frac{1}{m}\sum_j m_i m_j \frac{\delta_{ij} - a_i a_j}{a_i^2 a_j} \\ \frac{m_i}{a_i^3}(1-a_i^2) + \frac{1}{m^2}\sum_{j,k} m_i m_j m_k \frac{\delta_{jk} - a_j a_k}{a_i a_j a_k}\} = 0, \end{aligned} \tag{4.18}$$

where $1 \le \lambda \le r-1$. (4.18) is equivalent to

$$\begin{aligned} -\frac{m_i}{m a_i}\sum_j m_j(1-\frac{1}{a_j^2}) - \frac{1}{m}\sum_j m_i m_j \frac{\delta_{ij} - a_i a_j}{a_i^2 a_j} \\ \frac{m_i}{a_i^3}(1-a_i^2) + \frac{1}{m^2}\sum_{j,k} m_i m_j m_k \frac{\delta_{jk} - a_j a_k}{a_i a_j a_k} = 0. \end{aligned} \tag{4.19}$$

Therefore (4.19) becomes

$$-2m + \frac{m - m_i}{a_i^2} + \sum_j \frac{m_j}{a_j^2}(1 + \frac{m_j}{m}) = 0, \quad i = 1, \cdots, r. \tag{4.20}$$

We solve (4.20) to get

$$a_i^2 = \frac{m - m_i}{m(r-1)}, \qquad i = 1, \cdots, r.$$

We complete the proof of Proposition 4.1.

Remark 4.3. *When $r = 2$, Proposition 4.1 reduces to Theorem 3.1.*

Remark 4.4. *We also know that $S^{m_1}(a_1) \times \cdots \times S^{m_r}(a_r)$ in S^{m+r-1} is a m-dimensional minimal submanifold if and only if*

$$a_i = \sqrt{\frac{m_i}{m}}, \qquad \sum_i m_i = m, \quad i = 1, \cdots, r.$$

Thus it follows that $S^{m_1}(a_1) \times \cdots \times S^{m_r}(a_r)$ in S^{m+r-1} is an m-dimensional minimal and Willmore submanifold if and only if

$$m_1 = m_2 = \cdots = m_r = \frac{m}{r}, \quad a_i = \sqrt{\frac{1}{r}}.$$

Acknowledgements. The third author would like to thank Professor U. Simon for his hospitality during his research stay at the TU Berlin. We would like to thank the referee for his useful suggestions and comments.

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Eingegangen am 11. April 2001