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Hypersurfaces with constant scalar curvature in space forms★

Li Haizhong

Department of Applied Mathematics, Tsinghua University, Beijing, 100084, People's Republic of China

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1 Introduction and theorems

Let $R^{n+1}(c)$ be an $(n+1)$-dimensional Riemannian manifold with constant sectional curvature c, we also call it space form. When $c = 1$, $R^{n+1}(c) = S^{n+1}$ (i.e. $(n+1)$-dimensional unit sphere space); when $c = 0$, $R^{n+1}(c) = E^{n+1}$ (i.e. $(n+1)$-dimensional Euclidean space). Let M be an n-dimensional compact hypersurface in $R^{n+1}(c)$, and $e_1, \ldots, e_n$ a local orthonormal frame field on $M, \omega_1, \ldots, \omega_n$ its dual coframe field. Then the second fundamental form of M is

$$B = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \,. \tag{1.1}$$

Further, near any given point $p \in M$, we can choose a local frame field $e_1, \ldots, e_n$ so that at p, $\sum_{ij} h_{ij}\omega_i \otimes \omega_j = \sum_i k_i\omega_i \otimes \omega_j$, then the Gauss equation says

$$R_{ijij} = c + k_i k_j, \quad i \neq j \,. \tag{1.2}$$

In particular

$$n(n-1)(R-c) = n^2H^2 - |B|^2 \,, \tag{1.3}$$

where R is the normalized scalar curvature, $H = \frac{1}{n}\sum_i k_i$ the mean curvature and $|B|^2 = \sum_i k_i^2$ the norm square of the second fundamental form of M.

As well known, there are many rigidity results for minimal hypersurfaces or hypersurfaces with constant mean curvature H in $R^{n+1}(c)$, for example, see [S, NS, L] etc. Here a practical and powerful tool is to compute the Laplacian of some global geometric invariants on M, e.g., $|B|^2$, $f_s = \sum_i k_i^s$, $s \geqq 3$, etc., which method is first invented by J. Simons [S].

But unfortunately, above method does not work for the study of hyper surfaces with constant scalar curvature in $R^{n+1}(c)$. As far as we know, almost

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no rigidity results for this class of hypersurfaces until Cheng and Yau [CY] found the following famous result by the study of a new self-adjoint differential operator □ introduced by themselves (see [CY] or section 2 below):

Theorem 1 [CY]. *Let M be an n-dimensional compact hypersurface with constant normalized scalar curvature R in $R^{n+1}(c)$. If*
(1) $R - c \geqq 0$,
(2) *the sectional curvature K of M is non-negative, i.e. $K \geqq 0$,*
then M is either a totally umbilical hypersurface, or a (Riemannian) product of two totally umbilical constantly curved submanifolds.

In this paper, we will prove the following two rigidity theorems by the study of Cheng-Yau's self-adjoint operator □ and some new estimates

Theorem 2. *Let M be an n-dimensional ($n \geqq 3$) compact hypersurface with constant normalized scalar curvature R in an $(n+1)$-dimensional unit sphere S^{n+1}. If*
(1) $\bar{R} \equiv R - 1 \geqq 0$,
(2) *the norm square $|B|^2$ of the second fundamental form of M satisfies*

$$n\bar{R} \leqq |B|^2 \leqq \frac{n}{(n-2)(n\bar{R}+2)}[n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n], \tag{1.4}$$

then either

$$|B|^2 \equiv n\bar{R}, \tag{1.5}$$

and M is a totally umbilical hypersurface; or

$$|B|^2 \equiv \frac{n}{(n-2)(n\bar{R}+2)}[n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n], \tag{1.6}$$

and $M = S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, $r = \sqrt{\dfrac{n-2}{n(R+1)}}$ and $\bar{R} = R - 1$.

Theorem 3. *Let M be an n-dimensional ($n \geqq 3$) compact hypersurface with constant normalized scalar curvature R in an $(n+1)$-dimensional Euclidean space E^{n+1}. If the norm square $|B|^2$ of the second fundamental form of M satisfies*

$$nR \leqq |B|^2 \leqq \frac{n(n-1)}{n-2}R, \tag{1.7}$$

then $|B|^2 \equiv nR$, and M is an n-dimensional round sphere $S^n(r)$, $r = \sqrt{\dfrac{1}{R}}$.

Remark 1. In case $n = 2$, R is the Gauss curvature K. Theorem 1 is valid for this case.

Remark 2. When M is a compact embedded hypersurface in E^{n+1}, Theorem 3 holds without condition (1.7) (see [R]). When M is a compact embedded hypersurface in S^{n+1}, Theorem 2 holds without condition (1.4) (see [MR]) (in this case, M is a totally umbilical hypersurface).

2 Preliminaries

Let M be an n-dimensional compact hypersurface in $R^{n+1}(c)$. For any $p \in M$, we choose a local orthonormal frame $e_1, \ldots, e_n, e_{n+1}$ in $R^{n+1}(c)$ around p, so that $e_1, \ldots, e_n$ are tangent to M. Take the corresponding dual coframe $\{\omega_1, \ldots, \omega_n, \omega_{n+1}\}$. In this paper, we make the following convention on the range of indices:

$$1 \leqq A, B, C \leqq n+1; \qquad 1 \leqq i, j, k \leqq n .$$

The structure equation of $R^{n+1}(c)$ are

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} = -\omega_{BA} ,$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - c\omega_A \wedge \omega_B .$$

If we denote by the same letters to the restrictions of ω_A, ω_{AB} to M, we have

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} = -\omega_{ji} , \tag{2.1}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l , \tag{2.2}$$

where R_{ijkl} is the curvature tensor of the induced metric on M.

Restricted to $M, \omega_{n+1} = 0$, thus

$$0 = d\omega_{n+1} = \sum_i \omega_{n+1i} \wedge \omega_i , \tag{2.3}$$

from Cartan's lemma we can write

$$\omega_{in+1} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji} . \tag{2.4}$$

The quadratic form $B = \sum_{ij} h_{ij}\omega_i \otimes \omega_j$ is the second fundamental form of M. Gauss equation is

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + h_{ik}h_{jl} - h_{il}h_{jk} , \tag{2.5}$$

$$n(n-1)(R-c) = n^2H^2 - |B|^2 , \tag{2.6}$$

where R is the normalized scalar curvature, $H = \frac{1}{n} \sum_i h_{ii}$ the mean curvature and $|B|^2 = \sum_{ij} h_{ij}^2$ the norm square of the second fundamental form of M, respectively.

Codazzi equation is

$$h_{ijk} = h_{ikj} , \tag{2.7}$$

where the covariant derivative of the second fundamental form is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj} . \tag{2.8}$$

The second covariant derivative of h_{ij} is defined by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_m h_{mjk}\omega_{mi} + \sum_m h_{imk}\omega_{mj} + \sum_m h_{ijm}\omega_{mk} . \tag{2.9}$$

By exterior differentiation of (2.8), we can see that the following Ricci identities hold

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl} . \tag{2.10}$$

For a C^2-function f defined on M, the gradient and the hessian (f_{ij}) are defined by

$$df = \sum_i f_i\omega_i, \qquad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji} . \tag{2.11}$$

The Laplacian of f is defined by $\Delta f = \sum_i f_{ii}$.

Let $T = \sum_{ij} T_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on M, where

$$T_{ij} = nH\delta_{ij} - h_{ij} . \tag{2.12}$$

Following Cheng-Yau [CY], we introduce an operator $\square$ associated to T acting on any C^2-function f by

$$\square f = \sum_{i,j} T_{ij}f_{ij} = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij} , \tag{2.13}$$

since T_{ij} is divergence-free, it follows [CY] that the operator $\square$ is self-adjoint relative to the L^2 inner product of M, i.e.

$$\int_M f\square g = \int_M g\square f . \tag{2.14}$$

Near a given point $p \in M$, we choose an orthonormal frame field $\{e_1,\ldots,e_n\}$ and their dual frame field $\{\omega_1,\ldots,\omega_n\}$, so that $h_{ij} = k_i\delta_{ij}$ at p, we have the following computation by use of (2.13) and (2.6)

$$\begin{aligned}\square(nH) &= nH\Delta(nH) - \sum_i k_i(nH)_{ii} \\ &= \tfrac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i k_i(nH)_{ii} \\ &= \tfrac{1}{2}n(n-1)\Delta R + \tfrac{1}{2}\Delta|B|^2 - n^2|\nabla H|^2 - \sum_i k_i(nH)_{ii} . \end{aligned} \tag{2.15}$$

On the other hand, we have through a standard calculation by use of (2.7) and (2.10) (also see (2.8) of [CY])

$$\frac{1}{2}\Delta|B|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_i k_i(nH)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2 . \tag{2.16}$$

Putting (2.16) into (2.15), we have

$$\square(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2 . \tag{2.17}$$

3 Lemmas and estimates

Now we assume the normalized scalar curvature R of M is constant, then from (2.17) we have

$$\square(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2 . \tag{3.1}$$

From (2.5), we have $R_{ijij} = c + k_i k_j$, putting this into (3.1), we obtain

$$\square(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + nc|B|^2 - n^2H^2c - |B|^4 + nH\sum_i k_i^3 . \tag{3.2}$$

Let $\mu_i = k_i - H$ and $|Z|^2 = \sum_i \mu_i^2$, we have

$$\sum_i \mu_i = 0, \qquad |Z|^2 = |B|^2 - nH^2 , \tag{3.3}$$

$$\sum_i k_i^3 = \sum_i \mu_i^3 + 3H|Z|^2 + nH^3 . \tag{3.4}$$

From (3.2)–(3.4), we get

$$\square(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + |Z|^2(nc + nH^2 - |Z|^2) + nH\sum_i \mu_i^3 . \tag{3.5}$$

We need the following lemma due to Okumura (see [O])

Lemma 3.1 [O]. *The same notations as above, for $n \geqq 3$, we have*

$$-\frac{n-2}{\sqrt{n(n-1)}}|Z|^3 \leqq \sum_i \mu_i^3 \leqq \frac{n-2}{\sqrt{n(n-1)}}|Z|^3 , \tag{3.6}$$

and equality holds in (3.6) *if and only if at least* $(n-1)$ *of the* μ_i *are equal.*

Proof. We can get Lemma 3.1 by using the method of Lagrange's multipliers to find the critical points of $\sum_i \mu_i^3$ subject to the conditions: $\sum \mu_i = 0$, $\sum_i \mu_i^2 = |Z|^2$. We omit it here.

Combining (3.5) with (3.6), we obtain

$$\square(nH) \geqq |\nabla B|^2 - n^2|\nabla H|^2 + |Z|^2\left(nc + nH^2 - |Z|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||Z|\right) . \tag{3.7}$$

The following lemma essentially due to Cheng-Yau (see [CY])

Lemma 3.2. *Assume the normalized scalar curvature $R = constant$ and $R - c \geqq 0$, then*

$$|\nabla B|^2 \geqq n^2|\nabla H|^2 . \tag{3.8}$$

Proof. From (2.6),

$$n^2H^2 - \sum_{i,j} h_{ij}^2 = n(n-1)(R-c) .$$

Taking the covariant derivative of the above expression, and using the fact $R = constant$, we get

$$n^2HH_k = \sum_{i,j} h_{ij}h_{ijk} .$$

It follows that

$$\sum_k n^4H^2(H_k)^2 = \sum_k\left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 \leqq \left(\sum_{i,j} h_{ij}^2\right)\sum_{i,j,k} h_{ijk}^2 , \tag{3.9}$$

that is

$$n^4H^2|\nabla H|^2 \leqq |B|^2|\nabla B|^2 . \tag{3.10}$$

On the other hand, from $R-c \geqq 0$, we have $n^2H^2 - |B|^2 \geqq 0$. Thus

$$n^2H^2|\nabla H|^2 \leqq H^2|\nabla B|^2$$

and Lemma 3.2 follows.

For brevity, we write

$$\bar{R} = R - c\,, \tag{3.11}$$

by (2.6), we know

$$|Z|^2 = |B|^2 - nH^2 = \frac{n-1}{n}(|B|^2 - n\bar{R})\,, \tag{3.12}$$

note that $|B|^2 \geqq n\bar{R}$, and $|B|^2 \equiv n\bar{R}$ if and only if M is totally umbilical.

By use of (2.6), (3.12), Lemma 3.2, we get from (3.7)

$$\begin{aligned}\Box(nH) &\geqq \frac{n-1}{n}(|B|^2 - n\bar{R})\left[nc + 2nH^2 - |B|^2 - (n-2)|H|\sqrt{|B|^2 - n\bar{R}}\right]\\ &= \frac{n-1}{n}(|B|^2 - n\bar{R})\left[nc + 2(n-1)\bar{R} - \frac{n-2}{n}|B|^2\right.\\ &\qquad\left. - \frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + |B|^2)(|B|^2 - n\bar{R})}\right]\,.\end{aligned} \tag{3.13}$$

This is the key estimate in our paper which will help us to prove Theorem 2 and Theorem 3.

4 The proof of Theorem 2

In this case, $R^{n+1}(c) = S^{n+1}$, that is, $c = 1$, we have from (3.13)

$$\begin{aligned}\Box(nH) \geqq \frac{n-1}{n}(|B|^2 - n\bar{R})&\left[n + 2(n-1)\bar{R} - \frac{n-2}{n}|B|^2\right.\\ &\left. - \frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + |B|^2)(|B|^2 - n\bar{R})}\right]\,.\end{aligned} \tag{4.1}$$

It is a direct check that our assumption condition (1.4), i.e.

$$|B|^2 \leqq \frac{n}{(n-2)(n\bar{R}+2)}[n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n] \tag{4.2}$$

is equivalent to

$$\left(n + 2(n-1)\bar{R} - \frac{n-2}{n}|B|^2\right)^2 \geqq \frac{(n-2)^2}{n^2}(n(n-1)\bar{R} + |B|^2)(|B|^2 - n\bar{R})\,. \tag{4.3}$$

But it is clear from (4.2) that (4.3) is equivalent to

$$n + 2(n-1)\bar{R} - \frac{n-2}{n}|B|^2 \geqq \frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + |B|^2)(|B|^2 - n\bar{R})}\,. \tag{4.4}$$

Therefore the right hand of (4.1) is non-negative, we also have $\int_M \Box(nH)dv = 0$, since M is compact and the operator $\Box$ is self-adjoint. Thus either

$$|B|^2 \equiv n\bar{R}\,, \tag{4.5}$$

that is, $|B|^2 \equiv nH^2, M$ is a totally umbilical hypersurface; or

$$|B|^2 \equiv \frac{n}{(n-2)(n\bar{R}+2)}[n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n]\,. \tag{4.6}$$

In the latter case, we have $H \neq 0$ from (2.6) and $\bar{R} = R - 1 \geqq 0$, thus the equalities hold in (3.6) and (3.8), we follow that $k_i = constant$ for all i and $(n-1)$ of the k_i are equal. After renumeration if necessary, we can assume that

$$k_1 = k_2 = \cdots = k_{n-1},\quad k_1 \neq k_n,\quad k_i = const\,.$$

that is, M is a isoparametric hypersurface in S^{n+1} with two distinct principal curvatures, then $M = S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, now $k_1 = \cdots = k_{n-1} = \sqrt{1-r^2}/r, k_n = -r/\sqrt{1-r^2}$. From (2/6), it is easy to see $n(n-1)\bar{R} = (n-1)(n-2-nr^2)/r^2$, thus $r = \sqrt{\frac{n-2}{n(\bar{R}+1)}}$. We complete the proof of Theorem 2.

5 The proof of Theorem 3

In this case, $R^{n+1}(c) = R^{n+1}$, i.e., $c = 0$, we first note that the condition $\bar{R} = R - 0 \geqq 0$ automatically holds (in fact $R > 0$), the reason is the following: since M is compact, there exists a point $p_0 \in M$, $k_i(p_0) > 0$, $i = 1, \ldots, n$, thus $n(n-1)R = \sum_{i \neq j}(k_i k_j)(p_0) > 0$, but from the assumption $R = constant$, we conclude $R > 0$ on M.

Choosing $c = 0$ in (3.13), we have

$$\Box(nH) \geqq \frac{n-1}{n}(|B|^2 - n\bar{R})\left[2(n-1)\bar{R} - \frac{n-2}{n}|B|^2 - \frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + |B|^2)(|B|^2 - n\bar{R})}\right]. \tag{5.1}$$

From (1.7), we easily follow that the right hand side of (5.1) is non-negative, thus $\Box(nH) \geqq 0$. Through a same process of the proof of Theorem 2, we reach the M is a isoparametric hypersurface, thus $|B|^2 \equiv nR$ and M is an n-dimensional round sphere $S^n(r)$, $r = \sqrt{\frac{1}{R}}$. We complete the proof of Theorem 3.

References

[CY] S.Y. Cheng, S.T. Yau: Hypersurfaces with constant scalar curvature. Math. Ann. **225** (1977), 195–204

[L] H.Z. Li: A characterization of Clifford minimal hypersurfaces in S^4. To appear in Proc. Amer. Math. Soc.

[MR] S. Montiel, A. Ros: Compact hypersurfaces: The Alexandrov theorem for higher order mean curvatures. In: H.B. Lawson, K. Tenenblat (eds) Differential Geometry, a Symposium in honor of M. do Carmo, Pitman Monogr. vol. 52, pp. 279–296, 1991

[NS] K. Nomizu, B. Smyth: A formula of Simon's type and hypersurfaces. J. Differential Geom. **3** (1969), 367–377

[O] M. Okumura: Hypersurfaces and a pinching problem on the second fundamental tensor. Amer. J. Math. **96** (1974), 207–213

[R] A. Ros: Compact hypersurfaces with constant scalar curvature and a congruence theorem. J. Differential Geom. **27** (1988), 215–220

[S] J. Simons: Minimal varieties in Riemannian manifolds. Ann. of Math., (2) **88** (1968), 62–105

Note added in proof. Choosing $c = -1$ in (3.13) and making the same process of the proof of Theorem 2 and Theorem 3, we can obtain a rigidity theorem for n-dimensional hypersurfaces with constant scalar curvature in $(n+1)$-dimensional hyperbolic space $H^{n+1}(-1)$.