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SOME NONEXISTENCE THEOREMS ON STABLE MINIMAL SUBMANIFOLDS

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We prove that there exist no stable minimal submanifolds in some n -dimensional ellipsoids, which generalizes J. Simons' result about the unit sphere and gives a partial answer to Lawson–Simons' conjecture.

1. Introduction. In [S], J. Simons proved that there exist no stable minimal submanifolds in the n -dimensional unit sphere S^n . In this paper, we establish the following general results.

THEOREM 1. *Let N^n be an n -dimensional compact hypersurface in the $(n + 1)$ -dimensional Euclidean space \mathbb{R}^{n+1} . If the sectional curvature \bar{K} of N^n satisfies*

$$(1) \quad 1/2 < \bar{K} \leq 1,$$

then there exist no stable m -dimensional minimal submanifolds in N^n for each m with $1 \leq m \leq n - 1$.

Remark 1. If N^n is an n -dimensional unit hypersphere S^n in \mathbb{R}^{n+1} , then the sectional curvature \bar{K} of S^n is 1, and from Theorem 1 we deduce that there exist no stable m -dimensional minimal submanifolds in S^n for each m with $1 \leq m \leq n - 1$, which was proved by Simons [S].

THEOREM 2. *Let N^n be an n -dimensional ($n \geq 4$) compact submanifold in an $(n + p)$ -dimensional Euclidean space \mathbb{R}^{n+p} . Let R and H denote the normalized scalar curvature and the mean curvature functions of N^n , respectively. If R satisfies the following pointwise $n(n - 2)/(n - 1)^2$ -pinching condition:*

$$(2) \quad \frac{n(n - 2)}{(n - 1)^2} H^2 < R \leq H^2,$$

then there exist no stable m -dimensional minimal submanifolds in N^n for each m with $2 \leq m \leq n - 2$.

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COROLLARY 1. Let N^n be an n -dimensional ($n \geq 4$) compact hypersurface in \mathbb{R}^{n+1} . If all the principal curvatures k_a of N^n satisfy

$$(3) \quad 0 < k_a < \sqrt{\frac{1}{n(n-1)} \sum_{b=1}^n k_b}, \quad 1 \leq a \leq n,$$

then there exists no m -dimensional minimal submanifold in N^n for each m with $2 \leq m \leq n-2$.

As direct applications of Theorem 1 and Corollary 1, we have

PROPOSITION 1. Let N^n be the following n -dimensional ($n \geq 4$) ellipsoid in \mathbb{R}^{n+1} :

$$(4) \quad N^n : \frac{x_1^2}{a_1^2} + \dots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1, \quad 0 < a_1 \leq a_2 \leq \dots \leq a_{n+1},$$

(1) If $1 \leq a_{n+1} < \sqrt[3]{2}$ and $a_1 \geq \sqrt{a_{n+1}}$, then there exist no stable m -dimensional minimal submanifolds of N^n for each m with $1 \leq m \leq n-1$.

(2) If $a_{n+1}/a_1 < \sqrt[6]{n/(n-1)}$, then there exist no stable m -dimensional minimal submanifolds of N^n for each m with $2 \leq m \leq n-2$.

REMARK 2. It can be proved in a similar way that the above results all keep valid for stable m -currents on N^n (for concepts of stable current, see Lawson-Simons [LS]). For example, we can state the counterpart of Theorem 1 as follows:

THEOREM 1'. Let N^n be an n -dimensional compact hypersurface in the $(n+1)$ -dimensional Euclidean space \mathbb{R}^{n+1} . If the sectional curvature \bar{K} of N^n satisfies

$$(5) \quad 1/2 < \bar{K} \leq 1,$$

then there exist no stable m -currents on N^n for each m with $1 \leq m \leq n-1$.

REMARK 3. Let N^n be an n -dimensional compact hypersurface in \mathbb{R}^{n+1} and suppose that every principal curvature k_a of N^n satisfies $\sqrt{\delta} < k_a \leq 1$ ($a = 1, \dots, n$). H. Mori [M] and Y. Ohnita [O] proved the conclusion of Theorem 1' under the stronger conditions $\delta > n/(n+1)$ and $\delta > 1/2$, respectively. Our Theorem 1' also gives a partial answer to the following Lawson-Simons' conjecture:

CONJECTURE ([LS]). Let N^n be a compact n -dimensional connected Riemannian manifold with the sectional curvature \bar{K} satisfying

$$(6) \quad 1/4 < \bar{K} \leq 1.$$

Then there exist no stable m -currents on N^n for each m with $1 \leq m \leq n-1$.

We are greatly indebted to P. F. Leung's papers [L1, L2] which motivated us to do this work.

2. Basic formulas and notations. In this paper, we shall make use of the following convention on the ranges of indices:

$$\begin{aligned} 1 \leq A, B, C, \dots \leq n+p; \quad 1 \leq a, b, c, \dots \leq n; \quad n+1 \leq \mu, \nu, \dots \leq n+p; \\ 1 \leq i, j, k, \dots \leq m; \quad m+1 \leq \alpha, \beta, \gamma, \dots \leq n. \end{aligned}$$

Let M^m and N^n be Riemannian manifolds of dimension m and dimension n , respectively. Let M^m be an m -dimensional compact minimal submanifold of N^n , $n > m$. For any normal variation vector field $U = \sum_{\alpha} u_{\alpha} e_{\alpha}$ of M^m , the second variation of the volume is given by (see [S])

$$(7) \quad I(U, U) = \int_{M^m} \left[\sum_{\alpha, i} u_{\alpha i}^2 - \sum_{\alpha, \beta} (\sigma_{\alpha\beta} + \bar{R}_{\alpha\beta} u_{\alpha} u_{\beta}) \right] dv,$$

where $u_{\alpha i}$ are the covariant derivatives of u_{α} ,

$$(8) \quad \sigma_{\alpha\beta} = \sum_{i, j} h_{ij}^{\alpha} h_{ij}^{\beta},$$

$$(9) \quad \bar{R}_{\alpha\beta} = \sum_i \bar{R}_{\alpha i \beta i},$$

and h_{ij}^{α} are the components of the second fundamental form h of M^m in N^n .

Now let $x : N^n \rightarrow \mathbb{R}^{n+p}$ be an n -dimensional submanifold in the $(n+p)$ -dimensional Euclidean space \mathbb{R}^{n+p} . We choose a local field of orthonormal frames $e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p}$ in \mathbb{R}^{n+p} such that, restricted to N^n , the vectors e_1, \dots, e_n are tangent to N^n . Their dual coframe fields are $\omega_1, \dots, \omega_n, \omega_{n+1}, \dots, \omega_{n+p}$. Then we have

$$(10) \quad dx = \sum_a \omega_a e_a,$$

$$(11) \quad de_a = \sum_b \omega_{ab} e_b + \sum_{\mu, b} B_{ab}^{\mu} \omega_b e_{\mu},$$

$$(12) \quad de_{\mu} = - \sum_{a, b} B_{ab}^{\mu} \omega_b e_a + \sum_{\nu} \omega_{\mu\nu} e_{\nu},$$

and the second fundamental form of N^n in \mathbb{R}^{n+p} is

$$(13) \quad B = \sum_{a, b, \mu} B_{ab}^{\mu} \omega_a \otimes \omega_b \otimes e_{\mu}.$$

The Gauss equation of N^n in \mathbb{R}^{n+p} is

$$(14) \quad n(n-1)R = n^2 H^2 - S,$$

where R , H and S are the normalized scalar curvature, the mean curvature and the length square of the second fundamental form of N^n in \mathbb{R}^{n+p} , respectively.

3. An m -dimensional minimal submanifold in N^n . Let M^m be an m -dimensional minimal submanifold in N^n , and N^n be an n -dimensional submanifold in \mathbb{R}^{n+p} . In this case we can choose a local orthonormal basis $e_1, \dots, e_m, e_{m+1}, \dots, e_n, e_{n+1}, \dots, e_{n+p}$ in \mathbb{R}^{n+p} such that, restricted to M^m , the vectors e_1, \dots, e_m are tangent to M^m , e_1, \dots, e_n are tangent to N^n , e_{n+1}, \dots, e_{n+p} are normal to N^n . Their dual coframe fields are $\omega_1, \dots, \omega_m, \omega_{m+1}, \dots, \omega_n, \omega_{n+1}, \dots, \omega_{n+p}$. From (10)–(12), restricted to M^m , we have

$$(15) \quad dx = \sum_i \omega_i e_i,$$

$$(16) \quad de_i = \sum_j \omega_{ij} e_j + \sum_{\alpha,j} h_{ij}^\alpha \omega_j e_\alpha + \sum_{\mu,j} B_{ij}^\mu \omega_j e_\mu,$$

$$(17) \quad de_\alpha = - \sum_{i,j} h_{ij}^\alpha \omega_i e_j + \sum_\beta \omega_{\alpha\beta} e_\beta + \sum_{\mu,j} B_{\alpha j}^\mu \omega_j e_\mu,$$

$$(18) \quad de_\mu = - \sum_{i,j} B_{ij}^\mu \omega_i e_j - \sum_{\alpha,j} B_{\alpha j}^\mu \omega_j e_\alpha + \sum_\nu \omega_{\mu\nu} e_\nu,$$

where $h = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha$ is the second fundamental form of M^m in N^n and $\sum_i h_{ii}^\alpha = 0$ for any α , since M^m is a minimal submanifold in N^n .

We choose the following normal variation vector field of M^m in N^n :

$$(19) \quad U = \sum_\alpha u_\alpha e_\alpha, \quad u_\alpha = \langle \Lambda, e_\alpha \rangle,$$

where Λ is a constant vector in \mathbb{R}^{n+p} .

Using (15)–(18), a straightforward computation shows

$$(20) \quad u_{\alpha i} = - \sum_k h_{ki}^\alpha u_k + \sum_\mu B_{\alpha i}^\mu u_\mu,$$

$$(21) \quad \sum_{\alpha,i} u_{\alpha i}^2 = \sum_{\alpha,i} \left[\sum_{j,k} h_{ki}^\alpha h_{ij}^\alpha u_k u_j + \sum_{\mu,\nu} B_{\alpha i}^\mu B_{\alpha i}^\nu u_\mu u_\nu - 2 \sum_{\mu,k} h_{ki}^\alpha B_{\alpha i}^\mu u_k u_\mu \right],$$

where

$$(22) \quad u_j = \langle \Lambda, e_j \rangle, \quad u_\mu = \langle \Lambda, e_\mu \rangle.$$

Let E_1, \dots, E_{n+p} be a fixed orthonormal basis of \mathbb{R}^{n+p} , and $U_A = \sum_\alpha \langle E_A, e_\alpha \rangle e_\alpha$. Since

$$(23) \quad \sum_{A=1}^{n+p} \langle E_A, v \rangle \langle E_B, w \rangle = \langle v, w \rangle$$

for any vectors v, w in \mathbb{R}^{n+p} , putting (21) into (7) and using (22) and (23),

we obtain

$$\begin{aligned}
(24) \quad \text{trace}(I) &\equiv \sum_{A=1}^{n+p} I(U_A, U_A) \\
&= - \int_{M^m} \left[- \sum_{\alpha, k, \mu} (B_{\alpha k}^\mu)^2 + \sum_{\alpha} \bar{R}_{\alpha\alpha} \right] dv \\
&= - \int_{M^m} \sum_{\alpha, k} \left[- \sum_{\mu} (B_{\alpha k}^\mu)^2 + \bar{R}_{\alpha k \alpha k} \right] dv \\
&= - \int_{M^m} \left[- \sum_{\alpha, \mu, k} B_{\alpha\alpha}^\mu B_{kk}^\mu + 2 \sum_{\alpha, k} \bar{R}_{\alpha k \alpha k} \right] dv \\
&= \int_{M^m} \left[2 \sum_{\mu, \alpha, k} (B_{\alpha k}^\mu)^2 - \sum_{\mu, \alpha, k} B_{\alpha\alpha}^\mu B_{kk}^\mu \right] dv.
\end{aligned}$$

Thus we obtain

PROPOSITION 2. *Let N^n be an n -dimensional compact submanifold in \mathbb{R}^{n+p} . Let M^m be an m -dimensional compact minimal submanifold of N^n . If*

$$(25) \quad \text{trace}(I) = \int_{M^m} \left[2 \sum_{\mu, \alpha, k} (B_{\alpha k}^\mu)^2 - \sum_{\mu, \alpha, k} B_{\alpha\alpha}^\mu B_{kk}^\mu \right] dv < 0,$$

then M^m is not a stable minimal submanifold of N^n .

4. The proof of Theorem 1. Let N^n be an n -dimensional hypersurface in \mathbb{R}^{n+1} and M^m be an m -dimensional compact minimal submanifold in N^n . At a given point $p \in M^m$ in N^n , we can choose a local orthonormal frame field $e_1^*, \dots, e_n^*, \vec{n}$ in \mathbb{R}^{n+1} such that e_1^*, \dots, e_n^* are tangent to N^n and at $p \in M^m$,

$$(26) \quad B_{ab}^* = \langle B(e_a^*, e_b^*), \vec{n} \rangle = k_a \delta_{ab}, \quad 1 \leq a, b \leq n,$$

where the k_a are the principal curvatures of N^n in \mathbb{R}^{n+1} .

Since M^m is an m -dimensional compact minimal submanifold in N^n , at a given point $p \in M^m$ in N^n , we can also choose a local orthonormal frame field $e_1, \dots, e_m, e_{m+1}, \dots, e_n$ in N^n such that e_1, \dots, e_m are tangent to M^m . Noting that e_1, \dots, e_n and e_1^*, \dots, e_n^* are two local orthonormal frame fields in a neighborhood of $p \in M^m$, we can set

$$(27) \quad e_i = \sum_{b=1}^n A_i^b e_b^*, \quad 1 \leq i \leq m,$$

$$(28) \quad e_\alpha = \sum_{b=1}^n A_\alpha^b e_b^*, \quad m+1 \leq \alpha \leq n,$$

where $(A_a^b) \in SO(n)$, i.e.

$$(29) \quad \sum_{a=1}^n A_b^a A_c^a = \delta_{bc}, \quad \sum_{a=1}^n A_a^b A_a^c = \delta^{bc}.$$

It is a direct verification that at $p \in M^m$, by use of (26)–(29) and (1),

$$(30) \quad \begin{aligned} \sum_{\alpha,k} B_{\alpha\alpha} B_{kk} &= \sum_{\alpha,k} \langle B(e_\alpha, e_\alpha), B(e_k, e_k) \rangle \\ &= \sum_{\alpha,k,a,b,c,d} A_\alpha^a A_\alpha^b A_k^c A_k^d \langle B(e_a^*, e_b^*), B(e_c^*, e_d^*) \rangle \\ &= \sum_{\alpha,k,a,c} k_a k_c (A_\alpha^a)^2 (A_k^c)^2 \\ &= \sum_{a,c} \bar{R}_{acac} (A_\alpha^a)^2 (A_k^c)^2 \\ &\leq \sum_{a,c,\alpha,k} (A_\alpha^a)^2 (A_k^c)^2 = m(n-m), \end{aligned}$$

where $\bar{R}_{acac} = k_a k_c$ is the sectional curvature of N^n . From (1), we also have

$$(31) \quad -2 \sum_{\alpha,k} \bar{R}_{\alpha k \alpha k} < -2 \cdot \frac{1}{2} m(n-m) = -m(n-m).$$

Putting (30) and (31) into (24), we obtain $\text{trace}(I) < 0$. From Proposition 2, we infer that M^m is not a stable minimal submanifold of N^n .

5. The proof of Theorem 2. We first establish the following algebraic lemma in order to prove our Theorem 2:

LEMMA 1. *Let*

$$1 \leq a, b \leq n; \quad 1 \leq i, j \leq m; \quad m+1 \leq \alpha, \beta \leq n,$$

and consider the symmetric $n \times n$ matrix

$$\begin{bmatrix} T_{ij} & T_{i\alpha} \\ T_{\beta j} & T_{\beta\alpha} \end{bmatrix}$$

such that

$$(32) \quad \sum_{i=1}^m T_{ii} + \sum_{\alpha=m+1}^n T_{\alpha\alpha} = D, \quad \sum_{a,b=1}^n T_{ab}^2 = S.$$

Then:

(1) *If $m = 1$ or $m = n + 1$, we have*

$$(33) \quad \left(\sum_i T_{ii} \right)^2 - D \sum_i T_{ii} + 2 \sum_{i,\alpha} (T_{i\alpha})^2 \leq S + \frac{n-5}{2} D^2.$$

(2) If $2 \leq m \leq n - 2$, we have

$$(34) \quad \left(\sum_i T_{ii} \right)^2 - D \sum_i T_{ii} + 2 \sum_{i,\alpha} (T_{i\alpha})^2 \\ \leq \frac{m(n-m)}{n} S + \frac{|(2m-n)D|}{n^2} \sqrt{m(n-m)(Sn - D^2)} - \frac{2m(n-m)D^2}{n^2}.$$

Proof. We apply the Lagrange multiplier method to the problem (cf. P. F. Leung [L1, L2])

$$(35) \quad \left(\sum_i X_{ii} \right)^2 - D \sum_i X_{ii} + 2 \sum_{i,\alpha} (X_{i\alpha})^2 = \max!$$

subject to the constraints

$$(36) \quad \sum_i X_{ii} + \sum_\alpha X_{\alpha\alpha} = D$$

and

$$(37) \quad \sum_i (X_{ii})^2 + \sum_\alpha (X_{\alpha\alpha})^2 + 2 \sum_{i<j} (X_{ij})^2 + 2 \sum_{\alpha<\beta} (X_{\alpha\beta})^2 + 2 \sum_{i,\alpha} (X_{i\alpha})^2 = S,$$

where $S = \sum_{a,b} (T_{ab})^2$ and the X_{ab} form a symmetric $n \times n$ matrix

$$\begin{bmatrix} X_{ij} & X_{i\alpha} \\ X_{\beta j} & X_{\beta\alpha} \end{bmatrix}.$$

We consider the function

$$f = \left(\sum_i X_{ii} \right)^2 - D \sum_i X_{ii} + 2 \sum_{i,\alpha} (X_{i\alpha})^2 \\ + \lambda \left(\sum_i X_{ii} + \sum_\alpha X_{\alpha\alpha} - D \right) + \mu \left[\sum_i (X_{ii})^2 + \sum_\alpha (X_{\alpha\alpha})^2 \right. \\ \left. + 2 \sum_{i<j} (X_{ij})^2 + 2 \sum_{\alpha<\beta} (X_{\alpha\beta})^2 + 2 \sum_{i,\alpha} (X_{i\alpha})^2 - S \right],$$

where λ, μ are the Lagrange multipliers.

Differentiating with respect to each variable and equating to zero, we obtain

$$(38) \quad 2 \sum_j X_{jj} - D + \lambda + 2\mu X_{ii} = 0,$$

$$(39) \quad \lambda + 2\mu X_{\alpha\alpha} = 0,$$

$$(40) \quad 4X_{i\alpha} + 4\mu X_{i\alpha} = 0,$$

$$(41) \quad 4\mu X_{ij} = 0, \quad i < j,$$

$$(42) \quad 4\mu X_{\alpha\beta} = 0, \quad \alpha < \beta.$$

Hence (with the numbers standing for the corresponding left hand sides)

$$\sum_i X_{ii}(38) + \sum_\alpha X_{\alpha\alpha}(39) + \sum_{i,\alpha} X_{i\alpha}(40) + \sum_{i<j} X_{ij}(41) + \sum_{\alpha<\beta} X_{\alpha\beta}(42) = 0$$

gives

$$(43) \quad 2\left(\sum_i X_{ii}\right)^2 - D\sum_i X_{ii} + 4\sum_{i,\alpha} (X_{i\alpha})^2 = -(\lambda D + 2\mu S).$$

(1) Case $\mu = 0$. It is easy to see in this case

$$(44) \quad \left(\sum_i X_{ii}\right)^2 - D\sum_i X_{ii} + 2\sum_{i,\alpha} (X_{i\alpha})^2 = -\frac{D^2}{4}.$$

(2) Case $\mu = -1$. First we suppose $m(n-m) > n$, and putting $X_{\alpha\alpha} = \lambda/2$, $\sum_i X_{ii} = D - (n-m)\lambda/2$ into (38), we have

$$(45) \quad \lambda = \frac{(m-2)D}{m(n-m)-n}, \quad X_{ii} = \frac{(n-m-2)D}{2[m(n-m)-n]},$$

$$X_{\alpha\alpha} = \frac{(m-2)D}{2[m(n-m)-n]},$$

and

$$(46) \quad \left(\sum_i X_{ii}\right)^2 - D\sum_i X_{ii} + 2\sum_{i,\alpha} (X_{i\alpha})^2 = S - \frac{m(n-m)-4}{4[m(n-m)-n]}D^2$$

is another critical value.

Now suppose $m(n-m) = n$, i.e. $n = 4, m = 2$. If $\mu = -1$, then

$$(47) \quad X_{ii} = \frac{1}{2}(D - \lambda), \quad X_{\alpha\alpha} = \frac{\lambda}{2},$$

$$(48) \quad \left(\sum_i X_{ii}\right)^2 - D\sum_i X_{ii} + 2\sum_{i,\alpha} (X_{i\alpha})^2 = S - \frac{D^2}{2},$$

that is, equality holds in (34) in this case.

(3) Case $\mu \neq 0, -1$. Let $X = \sum_i X_{ii}$. Then

$$(49) \quad X_{\alpha\alpha} = -\frac{\lambda}{2\mu}, \quad 2\mu(X - D) = (n-m)\lambda,$$

$$(50) \quad \lambda = D - 2\left(1 + \frac{\mu}{m}\right)X.$$

Substituting (50) into the second formula of (49), we get

$$(51) \quad \mu = \frac{m(n-m)(D-2X)}{2(nX-mD)}, \quad \frac{\lambda}{\mu} = \frac{2}{n-m}(X-D).$$

From (43), we have

$$(52) \quad \frac{X(D - 2X)}{\mu} = \frac{\lambda}{\mu}D + 2S.$$

Putting (51) into (52), we get

$$X^2 - \frac{2mD}{n}X - \left(\frac{m(n-m)}{n}S - \frac{m}{n}D^2 \right) = 0,$$

that is,

$$(53) \quad X = \frac{m}{n}D \pm \sqrt{\frac{m(n-m)}{n} \left(S - \frac{D^2}{n} \right)}.$$

The critical value is

$$(54) \quad \left(\sum_i X_{ii} \right)^2 - D \sum_i X_{ii} + 2 \sum_{i,\alpha} (X_{i\alpha})^2 \\ = \frac{m(n-m)}{n}S + \frac{|(2m-n)D|}{n^2} \sqrt{m(n-m)(Sn - D^2)} - \frac{2m(n-m)D^2}{n^2}.$$

Hence, the critical values are

$$-\frac{D^2}{4}, \quad S - \frac{m(n-m) - 4}{4[m(n-m) - n]}D^2, \\ \frac{m(n-m)}{n}S + \frac{|(2m-n)D|}{n^2} \sqrt{m(n-m)(Sn - D^2)} - \frac{2m(n-m)D^2}{n^2}.$$

It can be verified directly by calculation that if $m = 1$ or $m = n - 1$, then $m(n - m) = n - 1$ and the maximum is $S + \frac{n-5}{4}D^2$; if $2 \leq m \leq n - 2$, the maximum is (cf. [L1])

$$\frac{m(n-m)}{n}S + \frac{|(2m-n)D|}{n^2} \sqrt{m(n-m)(Sn - D^2)} - \frac{2m(n-m)D^2}{n^2}.$$

This completes the proof of Lemma 1.

PROPOSITION 3. *Let N^n be an n -dimensional ($n \geq 4$) compact submanifold in \mathbb{R}^{n+p} . Let S be the length square of the second fundamental form. If*

$$(55) \quad S < 2nH^2 - |(2m-n)H| \sqrt{\frac{n}{m(n-m)}(S_H - nH^2)},$$

then there exist no stable m -dimensional minimal submanifolds of N^n for each m with $2 \leq m \leq n - 2$, where S_H is the length square of the second fundamental form in the direction of the mean curvature vector of N^n .

Proof. We choose a local orthonormal frame field e_1, \dots, e_{n+p} in \mathbb{R}^{n+p} with e_1, \dots, e_n tangent to N^n and e_{n+1}, \dots, e_{n+p} normal to N^n . Let e_{n+1}

be parallel to the mean curvature vector \vec{H} and

$$(56) \quad B(X, Y) = \sum_{\mu=n+1}^{n+p} B^\mu(X, Y)e_\mu,$$

then

$$(57) \quad \sum_a B^{n+1}(e_a, e_a) = nH, \quad \sum_a B^\mu(e_a, e_a) = 0, \quad n+2 \leq \mu \leq n+p.$$

Moreover,

$$(58) \quad \begin{aligned} & \sum_{i,\alpha} [2\|B(e_i, e_\alpha)\|^2 - \langle B(e_i, e_i), B(e_\alpha, e_\alpha) \rangle] \\ &= \left(\sum_i B^{n+1}(e_i, e_i) \right)^2 + 2 \sum_{i,\alpha} (B^{n+1}(e_i, e_\alpha))^2 - nH \sum_i B^{n+1}(e_i, e_i) \\ & \quad + \sum_{\mu=n+2}^{n+p} \left[\left(\sum_i B^\mu(e_i, e_i) \right)^2 + 2 \sum_{i,\alpha} (B^\mu(e_i, e_\alpha))^2 \right]. \end{aligned}$$

For each symmetric $n \times n$ -matrix $(B^{n+1}(e_a, e_b))$ and $(B^\mu(e_a, e_b))$, $1 \leq a, b \leq n$, $n+1 \leq \mu \leq n+p$, applying Lemma 1, we have

$$(59) \quad \begin{aligned} & \left(\sum_i B^{n+1}(e_i, e_i) \right)^2 + 2 \sum_{i,\alpha} (B^{n+1}(e_i, e_\alpha))^2 - nH \sum_i B^{n+1}(e_i, e_i) \\ & \leq \frac{m(n-m)}{n} S_H + |(2m-n)H| \sqrt{\frac{m(n-m)}{n} (S_H - nH^2) - 2m(n-m)H^2} \end{aligned}$$

and

$$(60) \quad \left(\sum_i B^\mu(e_i, e_i) \right)^2 + 2 \sum_{i,\alpha} (B^\mu(e_i, e_\alpha))^2 \leq \frac{m(n-m)}{n} \sum_{a,b} (B^\mu(e_a, e_b))^2.$$

Combining (58), (59) with (60), from assumption (55) we get

$$(61) \quad \begin{aligned} & \sum_{i,\alpha} [2\|B(e_i, e_\alpha)\|^2 - \langle B(e_i, e_i), B(e_\alpha, e_\alpha) \rangle] \\ & \leq \frac{m(n-m)}{n} S - 2m(n-m)H^2 + |(2m-n)H| \sqrt{\frac{m(n-m)}{n} (S_H - nH^2)} < 0. \end{aligned}$$

This completes the proof of Proposition 3.

Proof of Theorem 2. Let N^n be an n -dimensional ($n \geq 4$) compact submanifold in \mathbb{R}^{n+p} . By the Gauss equation (14) and the fact that $S \geq nH^2$, we know that condition (2) is equivalent to

$$(62) \quad S < \frac{n^2 H^2}{n-1}.$$

But (62) is equivalent to

$$(63) \quad \sqrt{S - nH^2} < \sqrt{\frac{n}{n-1}}|H| = \frac{1}{2}\sqrt{\frac{n}{n-1}}n|H| - \frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}|H|.$$

Now (63) is equivalent to

$$(64) \quad \left(\sqrt{S - nH^2} + \frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}|H| \right)^2 < \left(\frac{1}{2}\sqrt{\frac{n}{n-1}}n|H| \right)^2,$$

that is,

$$(65) \quad S < 2nH^2 - (n-2)\sqrt{\frac{n}{n-1}}|H|\sqrt{S - nH^2}.$$

Since $|2m-n|\sqrt{n/(m(n-m))} \leq (n-2)\sqrt{n/(n-1)}$ and $S_H \leq S$, we see that (65) implies (55) for each m with $2 \leq m \leq n-2$. Therefore, Theorem 2 follows from Proposition 3 directly.

6. The proof of Corollary 1 and Proposition 1

Proof of Corollary 1. Let N^n be an n -dimensional compact hypersurface in \mathbb{R}^{n+1} and let the principal curvatures be k_a , $1 \leq a \leq n$. By assumption (3), we have

$$(66) \quad S = \sum_i k_i^2 < \frac{n^2 H^2}{n-1}.$$

By the Gauss equation (14) and the fact $S \geq nH^2$, (66) is equivalent to (2). Now Corollary 1 follows from Theorem 2 directly.

Proof of Proposition 1. Let N^n be the following n -dimensional ($n \geq 4$) ellipsoid in \mathbb{R}^{n+1} :

$$N^n : \frac{x_1^2}{a_1^2} + \dots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1, \quad 0 < a_1 \leq a_2 \leq \dots \leq a_{n+1}.$$

It is not difficult to verify by a direct computation that the maximum and minimum of the principal curvatures are

$$k_{\max} = \frac{a_{n+1}}{a_1^2}, \quad k_{\min} = \frac{a_1}{a_{n+1}^2},$$

respectively.

(1) If $1 \leq a_{n+1} < \sqrt[3]{2}$ and $a_1 \geq \sqrt{a_{n+1}}$, then the sectional curvature \bar{K} of N^n satisfies

$$\frac{1}{2} < \frac{a_1^2}{a_{n+1}^4} = k_{\min}^2 \leq \bar{K} \leq k_{\max}^2 = \frac{a_{n+1}^2}{a_1^4} \leq 1.$$

Thus the conclusion of Proposition 1 follows from Theorem 1.

(2) If $a_{n+1}/a_1 < \sqrt[n]{n/(n-1)}$, then

$$k_\alpha - \sqrt{\frac{1}{n(n-1)}} \sum_{b=1}^n k_b \leq \frac{a_{n+1}}{a_1^2} - \sqrt{\frac{n}{n-1}} \frac{a_1}{a_{n+1}^2} < 0.$$

Thus the conclusion of Proposition 1 follows from Corollary 1.

7. Some remarks. Let N^n be an n -dimensional compact submanifold in an $(n+p)$ -dimensional unit sphere S^{n+p} and B the second fundamental form of N^n . By a reduction as in the proof of (24) (cf. (2.11) of Pan-Shen [PS]) we have

$$(67) \quad \text{trace}(I) = - \int_{M^m} \left[- \sum_{\alpha, k, \mu} (B_{\alpha k}^\mu)^2 + \sum_{\alpha} \bar{R}_{\alpha\alpha} \right] dv \\ = \int_{M^m} \left[-m(n-m) + 2 \sum_{\mu, \alpha, k} (B_{\alpha k}^\mu)^2 - \sum_{\mu, \alpha, k} B_{\alpha\alpha}^\mu B_{kk}^\mu \right] dv.$$

We can prove the following counterparts of Theorems 1 and 2 by making use of (67):

THEOREM 3. *Let N^n be an n -dimensional compact hypersurface in an $(n+1)$ -dimensional unit sphere S^{n+1} . If the sectional curvature \bar{K} of N^n satisfies*

$$(68) \quad 1/2 < \bar{K} \leq 1,$$

then there exist no stable m -dimensional minimal submanifolds in N^n for each m with $1 \leq m \leq n-1$.

THEOREM 4. *Let N^n be an n -dimensional ($n \geq 4$) compact submanifold in an $(n+p)$ -dimensional Euclidean sphere S^{n+p} . Let S and H be the length square of the second fundamental form and the mean curvature of N^n , respectively. If*

$$(69) \quad S < n + \frac{n^3}{2(n-1)} H^2 - \frac{n(n-2)}{2(n-1)} \sqrt{n^2 H^4 + 4(n-1) H^2},$$

then there exist no stable m -dimensional minimal submanifolds in N^n for each m with $2 \leq m \leq n-2$.

Remark 4. From the main theorem of [L2], we can prove that condition (2) or (69) implies $\text{Ric}(N^n) > 0$.

Remark 5. These conclusions keep valid for *stable currents* (see Lawson-Simons [LS] or Federer-Fleming [FF]).

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