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Hypersurfaces with Constant Scalar Curvature

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Introduction

Let M be a complete two-dimensional surface immersed into the three-dimensional Euclidean space. Then a classical theorem of Hilbert says that when the curvature of M is a non-zero constant, M must be the sphere. On the other hand, when the curvature of M is zero, a theorem of Hartman-Nirenberg [4] says that M must be a plane or a cylinder. These two theorems complete the classification of complete surfaces with constant curvature in R^3.

Generalizations of these theorems have been attempted by many authors (see the references in Kobayashi-Nomizu [5]). A typical theorem of this type is the following theorem of Thomas [7] which says that an Einstein hypersurface in R^{n+1} with $n \geqq 3$ is locally a sphere. As far as we know, all these theorems assume something on the Ricci tensor. In this paper, we propose to study complete hypersurfaces with constant scalar curvature. Like the theorems in two-dimension, this requires more global consideration.

When M is compact, we assume that the ambient manifold has constant sectional curvature c (and need not be complete). Suppose M has non-negative sectional curvature and constant scalar curvature $\geqq c$. Then we prove that either M is totally umbilical, the Riemannian product of two totally umbilical constantly curved submanifold, or M is flat. As a corollary, one sees that if the ambient manifold is R^{n+1}, M must be the sphere and if the ambient manifold is S^{n+1}, M has the form $S^{n-p} \times S^p$. In the course of the proof of this theorem, we introduce some self-adjoint differential operators which are interesting for their own right.

When M is complete and non-compact, we assume that M has non-negative curvature and the ambient space is R^{n+1}. In this case, we use the eigenvalue problem method initiated in the first author's thesis. We study the growth of the first eigenvalue of the operator mentioned above and compare this growth condition with the geometric equations that we derive. Our conclusion is that M must be a generalized cylinder $S^p \times R^{n-p}$.

The operators and the estimates that we have in this paper should have more applications. We hope we can come back to this again.

1. Second Order Differential Operators on a Riemannian manifold

Let $\{\omega_1, \ldots, \omega_n\}$ be a local orthonormal frame field defined on a Riemannian manifold M. Then the structure equations of M are given by

$$d\omega_i = \sum_j \omega_{ij} \Lambda \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{1.1}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \Lambda \omega_{kj} + \Omega_{ij}, \tag{1.2}$$

where

$$\Omega_{ij} = -\tfrac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \Lambda \omega_l \tag{1.3}$$

and

$$R_{ijkl} + R_{ijlk} = 0. \tag{1.4}$$

For any C^2-function f defined on M, we define its gradient and hessian by the following formulas

$$df = \sum_i f_i \omega_i, \tag{1.5}$$

$$\sum_j f_{ij} \omega_j = df_i + \sum_j f_j \omega_{ji}. \tag{1.6}$$

Let $\phi = \sum_{i,j} \phi_{ij} \omega_i \otimes \omega_j$ be a symmetric tensor defined on M. Then we can define an operator associated to ϕ by

$$\Box f = \sum_{i,j} \phi_{ij} f_{ij}. \tag{1.7}$$

The first observation is the following:

Proposition 1. *Let M be a compact orientable Riemannian manifold. Then the operator $\Box$ is self-adjoint iff*

$$\sum_j \phi_{ijj} = 0 \tag{1.8}$$

for all i.

Proof. Note that the covariant derivative of ϕ_{ij} is defined by

$$\sum_k \phi_{ijk} \omega_k = d\phi_{ij} + \sum_k \phi_{kj} \omega_{ki} + \sum_k \phi_{ik} \omega_{kj} \tag{1.9}$$

so that the condition (1.8) is independent of the choice of the frame field.

From condition (1.8), one verifies that for any C^2-functions f and g, we have

$$\int_M (\Box f) g = \int_M d\left(\sum_{i,j} \phi_{ij} f_i g * \omega_j\right) - \int_M \sum_{i,j} \phi_{ij} f_i g_j. \tag{1.10}$$

Hence, by Stokes' theorem,

$$\int_M (\Box f) g = \int_M f (\Box g). \tag{1.11}$$

Conversely, it is straightforward to see that the validity of (1.11) for all f and g implies (1.8).

Based on Proposition 1, let us give two examples of self-adjoint operators.

Let $R_{ij}=\sum_k R_{ikjk}$ be the Ricci tensor of the Riemannian manifold. Then we claim that the operator $\Box f=\sum_{i,j}\left(\frac{R}{2}\delta_{ij}-R_{ij}\right)f_{ij}$ is self-adjoint, where $R=\sum_i R_{ii}$ is the scalar curvature.

In fact, by the Bianchi identity,

$$\begin{aligned}\sum_j R_{ij,j}&=\sum_{k,j} R_{ikjk,j}\\ &=\sum_{k,j} R_{jkik,j}\\ &=-\sum_{k,j} R_{jkkj,i}-\sum_{k,j} R_{jkji,k}\\ &=R_i-\sum_{k,j} R_{jijk,k}\\ &=R_i-\sum_k R_{ik,k}\,. \end{aligned}\tag{1.12}$$

Hence $\sum_j R_{ij,j}=\frac{1}{2}R_i$ and our claim is proved.

Our second example is provided by $\phi_{ij}=\left(\sum_k \psi_{kk}\right)\delta_{ij}-\psi_{ij}$ where ψ_{ij} is a symmetric tensor satisfying the "Codazzi equation":

$$\psi_{ij,k}=\psi_{ik,j}\,.\tag{1.13}$$

Using (1.13), it is straightforward to check that $\Box f=\sum_{i,j}\phi_{ij}f_{ij}$ is self-adjoint.

The interesting second order elliptic operators are the elliptic ones. A sufficient condition for the operator in the first example to be elliptic is that the Ricci curvature of M is positive and pinched in the following sense: The ratio of the maximal Ricci curvature and the minimal Ricci curvature is less than $n-1$. We hope that this operator will be useful in intrinsic geometry.

2. Laplacian of a Symmetric Tensor

Let $\sum_{i,j}\phi_{ij}\omega_i\otimes\omega_j$ be a symmetric tensor defined on M. Then following Bochner, Calabi, Simons, Chern [1, 6, 3], we shall compute the Laplacian of this tensor.

The covariant derivative of ϕ_{ij} is defined by (1.9). The second covariant derivative of ϕ_{ijk} is defined by

$$\sum_l \phi_{ijkl}\omega_l=d\phi_{ijk}+\sum_m \phi_{mjk}\omega_{mi}+\sum_m \phi_{imk}\omega_{mj}+\sum_m \phi_{ijm}\omega_{mk}\,.\tag{2.1}$$

Exterior differentiate (1.9), we obtain

$$\sum_{l,k}\phi_{ijkl}\omega_l\Lambda\omega_k=\sum_k \phi_{kj}\Omega_{ki}+\sum_k \phi_{ik}\Omega_{kj}\,.\tag{2.2}$$

Therefore,

$$\phi_{ijkl}-\phi_{ijlk}=-\sum_m \phi_{mj}R_{milk}-\sum_m \phi_{im}R_{mjlk}\,. \tag{2.3}$$

The Laplacian of the tensor ϕ_{ij} is defined to be $\sum_k \phi_{ijkk}$ and so

$$\begin{aligned}\Delta\phi_{ij}&=\sum_k \phi_{ijkk}\\ &=\sum_k(\phi_{ijkk}-\phi_{ikjk})+\sum_k(\phi_{ikjk}-\phi_{ikkj})+\sum_k(\phi_{ikkj}-\phi_{kkij})+\Big(\sum_k\phi_{kk}\Big)_{ij}\\ &=\sum_k(\phi_{ijkk}-\phi_{ikjk})+\sum_k(\phi_{ikkj}-\phi_{kkij})+\Big(\sum_k\phi_{kk}\Big)_{ij}\\ &\quad-\sum_{m,k}\phi_{mk}R_{mikj}-\sum_{m,k}\phi_{im}R_{mkkj}\,.\end{aligned} \tag{2.4}$$

For tensors satisfying "Codazzi equation"

$$\phi_{ijk}=\phi_{ikj} \tag{2.5}$$

have

$$\Delta\phi_{ij}=\Big(\sum_k\phi_{kk}\Big)_{ij}-\sum_{m,k}\phi_{mk}R_{mikj}-\sum_{m,k}\phi_{im}R_{mkkj}\,. \tag{2.6}$$

Let $|\phi|^2=\sum_{i,j}\phi_{ij}^2$ and $\operatorname{tr}\phi=\sum_i\phi_{ii}$. Then equation (2.6) shows:

$$\tfrac{1}{2}\Delta|\phi|^2=\sum_{i,j,k}\phi_{ijk}^2+\sum_{i,j}\phi_{ij}(\operatorname{tr}\phi)_{ij}-\sum_{i,j,m,k}\phi_{ij}\phi_{mk}R_{mikj}-\sum_{i,j,m,k}\phi_{ij}\phi_{im}R_{mkkj}\,. \tag{2.7}$$

Choose a frame field $\{\omega_1,\dots,\omega_n\}$ so that $\phi_{ij}=\lambda_i\delta_{ij}$. Then (2.7) simplifies to

$$\tfrac{1}{2}\Delta|\phi|^2=\sum_{i,j,k}\phi_{ijk}^2+\sum_i\lambda_i(\operatorname{tr}\phi)_{ii}+\tfrac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2\,. \tag{2.8}$$

Denoting the second symmetric function of ϕ_{ij} by m, we have

$$\begin{aligned}m&=\sum_{i\neq j}\lambda_i\lambda_j\\ &=(\operatorname{tr}\phi)^2-|\phi|^2\,.\end{aligned} \tag{2.9}$$

Putting (2.9) and (2.8) together, we obtain

$$\tfrac{1}{2}\Delta(\operatorname{tr}\phi)^2=\Delta m+\sum_{i,j,k}\phi_{ijk}^2+\sum_i\lambda_i(\operatorname{tr}\phi)_{ii}+\tfrac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2\,. \tag{2.10}$$

Let $\Box f=\sum_{i,j}((\operatorname{tr}\phi)\delta_{ij}-\phi_{ij})f_{ij}$ be the self-adjoint operator defined in Section 1. Then (2.10) takes the form

$$\Box(\operatorname{tr}\phi)=\Delta m+\sum_{i,j,k}\phi_{ijk}^2-\sum_i(\operatorname{tr}\phi)_i^2+\tfrac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2\,. \tag{2.11}$$

Since $\Box$ is self-adjoint, we conclude that

$$0\geqq\int\Big[\sum_{i,j,k}\phi_{ijk}^2-\sum_i(\operatorname{tr}\phi)_i^2\Big]+\int\tfrac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2\,. \tag{2.12}$$

By Schwarz inequality,

$$\sum_{i,j,k} \phi_{ijk}^2 \geqq \frac{1}{|\phi|^2} \sum_k \Big(\sum_{i,j} \phi_{ij}\phi_{ijk} \Big)^2 . \tag{2.13}$$

Differentiating (2.9) and using (2.13) we obtain

$$\sum_{i,j,k} \phi_{ijk}^2 \geqq |\phi|^{-2} \sum_k [(\operatorname{tr}\phi)(\operatorname{tr}\phi)_k - \tfrac{1}{2} m_k]^2$$

$$\geqq |\phi|^{-2}[(1-\varepsilon)(\operatorname{tr}\phi)^2 \sum_k (\operatorname{tr}\phi)_k^2 + \left(\frac{1}{4} - \frac{1}{4\varepsilon}\right) \sum_k m_k^2] \tag{2.14}$$

for all $\varepsilon > 0$.

Substituting (2.14) into (2.12) we derive that

$$0 \geqq \int \left[(1-\varepsilon)(|\phi|^2 + m)|\phi|^{-2} \sum_k (\operatorname{tr}\phi)_k^2 - \sum_k (\operatorname{tr}\phi_k)^2 + \left(\frac{1}{4} - \frac{1}{4\varepsilon}\right) |\phi|^{-2} \sum_k m_k^2 \right]$$

$$+ \int \tfrac{1}{2} \sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 . \tag{2.15}$$

If m = constant, we can take $\varepsilon = 0$ in (2.15). Hence

$$0 \geqq \int_M \left[m|\phi|^{-2} \sum_k (\operatorname{tr}\phi)_k^2 + \tfrac{1}{2} \sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 \right] . \tag{2.16}$$

When $m \geqq 0$ and $R_{ijij} \geqq 0$, the integrand of the integral in (2.16) must be identically zero. Therefore (2.13) becomes equality and there are numbers c_k such that

$$\phi_{ijk} = c_k \phi_{ij} \tag{2.17}$$

when $|\phi|^2 \neq 0$.

Since we assume $\phi_{ij} = \lambda_i \delta_{ij}$ and $\phi_{ijk} = \phi_{ikj}$, it is easy to see that the only non-zero terms of ϕ_{ijk} and the terms ϕ_{iii}. Thus

$$(\phi_{ii} - \phi_{jj})\omega_{ij} = 0 . \tag{2.18}$$

Using the fact $\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 = 0$, we can then prove that M is the closure of $\bigcup_k 0_k$ where each point of the open set 0_k has a product neighborhood $N_1 \times N_2 \times \ldots \times N_l$ where the tangent space of each N_i is spanned by eigenvectors of ϕ_{ij} with the same eigenvalue.

Theorem 1. *Let $\sum_{i,j} \phi_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on a compact Riemannian manifold with non-negative sectional curvature. Suppose the Codazzi equation $\phi_{ij,k} = \phi_{ik,j}$ is satisfied. Let $\{\omega_i\}$ be a co-frame which diagonalizes the tensor ϕ_{ij} so that $\sum_{i,j} \phi_{ij}\omega_i \otimes \omega_j = \sum_i \lambda_i \omega_i \otimes \omega_i$. Then when $m = \sum_{i \neq j} \lambda_i \lambda_j$ is a non-negative constant, we have $m|\nabla \operatorname{tr}\phi| = 0$ and $\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 = 0$. Furthermore M is the closure of $\bigcup_i 0_i$ where each point of the open set 0_i has a product neighborhood $N_1 \times \ldots \times N_l$ so that the cotangent space of each N_i are spanned by the ω_k's with the same λ_k. In particular, when M is locally irreducible, all the eigenvalues of ϕ_{ij} are the same.*

3. Compact Hypersurfaces

The applications that we have in mind are of course the submanifold problems. Thus, let M be a compact hypersurface in a Riemannian manifold with constant sectional curvature c. Then the second fundamental form $\sum_{i,j} h_{ij}\omega_i \otimes \omega_j$ of M satisfies the Codazzi equation. Furthermore when $\{e_1, \dots e_n\}$ diagonalizes the second fundamental form so that $\sum_{i,j} h_{ij}\omega_i \otimes \omega_j = \sum_i \lambda_i \omega_i \otimes \omega_i$, then the Gauss equation says

$$R_{ijij} = c + \lambda_i \lambda_j . \tag{3.1}$$

In particular

$$\sum_{i \neq j} \lambda_i \lambda_j = n(n-1)(R-c) \tag{3.2}$$

where R is the normalized scalar curvature of M.

Therefore when R is constant and $\geqq c$, Theorem 1 applies. In this case, we shall exhaust the possibility of M.

Theorem 1 shows that when $\lambda_i \neq \lambda_j$

$$c + \lambda_i \lambda_j = 0 . \tag{3.3}$$

It is a simple algebraic fact that (3.3) implies there are at most two distinct λ_i's. Call them λ_1 and λ_2.

When $c=0$, either $\lambda_1 \equiv \lambda_2$ in which case M is totally umbilical or at some point of M, we have $\lambda_1 \lambda_2 = 0$ and $\lambda_1^2 + \lambda_2^2 \neq 0$. In the latter case, if the multiplicity of the zero eigenvalue is not equal to $\dim M - 1$, Equation (3.2) shows that the non-zero eigenvalue is a constant depending only on R (which implies the constancy of the multiplicity of this eigenvalue). From these data, we conclude, using (3.1), that either M is totally umbilical, a flat manifold, or the product of a totally umbilical, constantly curved submanifold with a totally geodesic flat manifold. (By totally umbilical submanifold, we mean a submanifold such that the principle curvatures are all equal with respect to any normal direction.)

When $c > 0$, $\lambda_1 \neq \lambda_2$ and $R \geqq c > 0$; (3.3) implies $\dim M \geqq 3$. One then uses (3.1) and (3.2) to show that both λ_i are constants and have constant multiplicities. Thus M is either a totally umbilical hypersurface or the product of two totally umbilical constantly curved submanifolds.

When $c < 0$, we have $R - c > 0$ and Theorem 1 shows that $\operatorname{tr}(h_{ij}) = \text{constant}$. Equation (3.2) again shows that M is either totally umbilical or the product of two totally umbilical constantly curved submanifolds.

Theorem 2. *Let M be a compact hypersurface with non-negative sectional curvature immersed in a manifold with constant sectional curvature c. Suppose the normalized scalar curvature of M is constant and greater than or equal to c. Then M is either totally umbilical, a (Riemannian) product of two totally umbilical constantly curved submanifolds or possibly a flat manifold which is different from the above two types. The last case can happen only if $c=0$. (If the ambient manifold is the Euclidean space, the last two cases cannot occur because of the compactness of M.)*

In Theorem 2, we assume the scalar curvature R is constant. Let us now relax this condition and get an estimate of the mean curvature of M in terms of the intrinsic geometry of M.

Suppose $nH\delta_{ij}-h_{ij}$ is positive semi-definite so that $\square f=\sum_{i,j}(nH\delta_{ij}-h_{ij})f_{ij}$ is a (possibly degenerate) elliptic operator.

Hence when nH attains its maximum at some point $\varrho\in M$, we see from (2.11) that at ϱ,

$$\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2\leqq -2n(n-1)\Delta R\,. \tag{3.4}$$

In particular,

$$\begin{aligned}\left(\min_{i\neq j}R_{ijij}\right)(nH)^2&\leqq\left(\min_{i\neq j}R_{ijij}\right)n\left(\sum_i\lambda_i^2\right)\\&\leqq\left(\min_{i\neq j}R_{ijij}\right)\left[\frac{n}{n-1}\,\frac{1}{2}\sum_{i,j}(\lambda_i-\lambda_j)^2+n^2(R-c)\right]\\&\leqq -n^2\Delta R+n^2(R-c)\min_{i\neq j}R_{ijij}\,.\end{aligned} \tag{3.5}$$

Since (3.5) holds at the point where nH achieves its maximum, we have proved the following

Theorem 3. *Let M be a compact hypersurface in a manifold with constant sectional curvature c. Suppose M has positive sectional curvature and the form $(nH\delta_{ij}-h_{ij})$ is positive semi-definite. Then*

$$\sup_M nH\leqq\sup_M\left\{-n^2\left[\min_{i\neq j}(R_{ijij})\right]^{-1}\Delta R+n^2(R-c)\right\}^{\frac{1}{2}} \tag{3.6}$$

and hence the length of the second fundamental form of M can be estimated in terms of quantities defined intrinsically.

4. Noncompact Convex Hypersurfaces

Let M be a complete hypersurface with non-negative sectional curvature in the Euclidean space. Then we can choose the normal of M so that its second fundamental form is positive semi-definite.

As in Section 3, we consider the operator $\square f=\sum_{i,j}(nH\delta_{ij}-h_{ij})f_{ij}$ where H is the mean curvature and h_{ij} is the second fundamental form.

Let X and e_{n+1} be the position vector and the normal vector of M respectively. Then we shall compute $\square X$ and $\square e_{n+1}$.

Let $\{e_1,\ldots,e_n\}$ be a local orthonormal frame field of M. Then

$$X_{ij}=h_{ij}e_{n+1} \tag{4.1}$$

for all i,j.

Hence by (3.2),

$$\begin{aligned}\square X&=\sum_{i,j}(nH\delta_{ij}-h_{ij})h_{ij}e_{n+1}\\&=\left((nH)^2-\sum_{i,j}h_{ij}^2\right)e_{n+1}\\&=n(n-1)R\,e_{n+1}\,.\end{aligned} \tag{4.2}$$

To compute $\square e_{n+1}$, we have

$$d e_{n+1} = -\sum_{i,k} h_{ik} e_i \omega_k, \tag{4.3}$$

$$d\Big(-\sum_i h_{ik} e_i\Big) - \sum_l h_{il} e_i \omega_{lk} = \sum_l \Big(-\sum_i h_{ik} h_{il} e_{n+1} - \sum_i h_{ikl} e_i\Big)\omega_l, \tag{4.4}$$

$$\begin{aligned}
\square e_{n+1} &= \sum_{k,l} (nH\delta_{kl} - h_{kl})\Big(-\sum_i h_{ik} h_{il} e_{n+1} - \sum_i h_{ikl} e_i\Big) \\
&= -\sum_{k,l} (nH\delta_{kl} - h_{kl})\Big(\sum_i h_{ik} h_{il}\Big) e_{n+1} \\
&\quad - \sum_{i,k,l} (nH\delta_{kl} - h_{kl}) h_{ikl} e_i \\
&= -\sum_{k,l} (nH\delta_{kl} - h_{kl})\Big(\sum_i h_{ki} h_{il}\Big) e_{n+1} \\
&\quad - \sum_i \Big[nH(nH)_i - \sum_{k,l} h_{ikl} h_{kl}\Big] e_i \\
&= -\sum_{k,l} (nH\delta_{kl} - h_{kl})\Big(\sum_i h_{ki} h_{il}\Big) e_{n+1} \\
&\quad - \tfrac{1}{2} n(n-1) \sum_i R_i e_i
\end{aligned} \tag{4.5}$$

where (3.2) has been used in the last equality.

We shall make use of the formulas (4.4) and (4.5) to deal with our problem. First of all, we note the following

Proposition 2. *Let $\square$ be a formally self-adjoint second order (possibly degenerate) elliptic operator defined on a compact manifold M with boundary. Let f be a C^2 positive function. Then for any non-negative C^2-function g such that $g|\partial M = 0$, we have*

$$\Big(-\int_M g\,\square g\Big)\Big(\int_M g^2\Big)^{-1} \geqq \inf_M \Big(\frac{-\square f}{f}\Big). \tag{4.6}$$

Proof. We note that we have only to prove (4.6) by assuming $\square$ is non-degenerate elliptic. In fact, one can simply replace $\square$ by $\square + \varepsilon\Delta$ and let $\varepsilon \to 0$.

Let λ be the first eigenvalue and g_λ be the first eigenfunction of $\square$ over D with the boundary condition $g_\lambda|\partial D = 0$. Then it is well-known that the left hand side of (4.6) is always not less than λ and g_λ is positive in the interior of D.

Consider the function g_λ/f defined on D. Then at the point where g_λ/f attains its maximum, one can verify easily that $\lambda = \dfrac{-\square g_\lambda}{g_\lambda} \geqq \dfrac{-\square f}{f}$. This proves (4.6).

Theorem 4. *Let M be a complete non-compact hypersurface in the Euclidean space with non-negative curvature. Suppose the scalar curvature of M is constant, then M is a generalized cylinder $S^{n-p} \times R^p$.*

Proof. Since M is convex, there is a unit vector a in the Euclidean space so that $\langle e_{n+1}, a\rangle \geqq 0$ on M (see Wu [8]). If $\langle e_{n+1}, a\rangle = 0$ at one point, then we claim that $\langle e_{n+1}, a\rangle$ is identically zero.

First of all, we note that if the scalar curvature of M is non-zero, the operator $\square$ is elliptic. In fact $\square$ being degenerate elliptic means that at some point in M, $\sum_{i \neq j} \lambda_i$ is equal to zero for some principle curvature λ_j. This implies $\lambda_i = 0$ for all $i \neq j$ and the scalar curvature is zero at that point. When the scalar curvature of M is zero, M is flat and Theorem 4 follows from Hartman-Nirenberg [4].

Our claim now follows by applying the minimal principle to the elliptic equation

$$\square \langle e_{n+1}, a \rangle = -\sum_{k,l} (nH\delta_{kl} - h_{kl}) \left(\sum_i h_{ki} h_{il} \right) \langle e_{n+1}, a \rangle . \tag{4.7}$$

Therefore, we conclude that either $\langle e_{n+1}, a \rangle$ is everywhere positive or $\langle e_{n+1}, a \rangle \equiv 0$. In the latter case, we can split out one line and continue by induction to prove the theorem (see [2]). Hence we can apply the proposition to (4.7) and deduce

$$\left(-\int_D g \square g \right) \left(\int_D g^2 \right)^{-1} \geqq \min_D \sum_{k,l} (nH\delta_{kl} - h_{kl}) \left(\sum_i h_{ki} h_{il} \right) \tag{4.8}$$

for all smooth function g with compact support D.

We shall apply g to a function defined by $\langle X, a \rangle$. Since M is essentially a graph along a (see Wu [8]), the set $D_r = \{X | \langle X, a \rangle \leqq r\}$ is compact for all $r > 0$. We can then apply $r - \langle X, a \rangle$ to (4.8) with D replaced by D

On the other hand, applying (4.2), we have

$$\begin{aligned} -&\left[\int_{D_r} (r - \langle X, a \rangle) \square (r - \langle X, a \rangle) \right] \left(\int_{D_r} (r - \langle X, a \rangle)^2 \right)^{-1} \\ &\leqq n(n-1) r R \left(\int_{D_r} \langle e_{n+1}, a \rangle \right) \left[\frac{r^2}{4} \operatorname{Vol}(D_{r/2}) \right]^{-1} \\ &= 4n(n-1) R r^{-1} \operatorname{Vol}(D_r) [\operatorname{Vol}(D_{r/2})]^{-1} . \end{aligned} \tag{4.9}$$

Since M is convex, it is easy to verify that there are constants c_1 and c_2 so that $\operatorname{Vol}(D_r) \leqq c_1 r^n + c_2$. [For example, one sees that $\langle X, a \rangle$ is asymptotic to the geodesic distance of M and $\operatorname{Vol}(D_r)$ is asymptotically the volume of the geodesic hall.] This implies that $\lim_{r \to \infty} \inf r^{-\varepsilon} \operatorname{Vol}(D_r) [\operatorname{Vol}(D_{r/2})]^{-1} = 0$, for any $\varepsilon > 0$.

Combining (4.8), (4.9) and the above information, we conclude that

$$\inf_M \sum_{k,l} (nH\delta_{kl} - h_{kl}) \left(\sum_i h_{ki} h_{il} \right) = 0 . \tag{4.10}$$

Let λ_1 and λ_2 be principle curvatures such that $\lambda_1 \lambda_2 \geqq R$. Then

$$\begin{aligned} &\sum_{k,l} (nH\delta_{k,l} - h_{kl}) \left(\sum_i h_{ki} h_{il} \right) \\ &\quad \geqq \lambda_1 \lambda_2^2 + \lambda_2 \lambda_1^2 \\ &\quad \geqq (\lambda_1 \lambda_2)(\lambda_1 + \lambda_2) \\ &\quad \geqq \lambda_1 \lambda_2 \sqrt{2\lambda_1 \lambda_2} \\ &\quad \geqq \sqrt{2} R^{3/2} . \end{aligned} \tag{4.11}$$

Since R is constant, we conclude from (4.10) and (4.11) that $R = 0$ and Theorem 4 again follows from Hartman-Nirenberg [4].

References

1. Calabi, E.: Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. Mich. Math. J. **5**, 105 (1958)
2. Cheng, S. Y., Yau, S. T.: Differential equations on Riemannian manifolds and their geometric applications (to appear in Commun. Pure Appl. Math.)
3. Chern, S. S.: Minimal submanifolds in a Riemannian manifold. Mimeographed lecture notes. Univ. of Kansas 1968
4. Hartman, P., Nirenberg, L.: On spherical image maps whose Jacobians do note change sign. Amer. J. Math. **81**, 901 (1959)
5. Kobayashi, S., Nomizu, K.: Foundations of differential geometry. Vol. II. New York: Wiley-Interscience 1969
6. Simons, J.: Minimal varieties in riemannian manifolds. Ann. Math. **88**, 62 (1968)
7. Thomas, T. Y. T.: On closed spaces of constant mean curvature. Amer. J. Math. **58**, 702 (1936)
8. Wu, H.: The spherical images of convex hypersurfaces. J. Diff. Geom. **9**, 279 (1974)

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