A note on Hopf's conjecture

Oscar Perdomo

Universidad del Valle, Cali - Colombia

ABSTRACT: Let (S_i, g_i), $i = 1, 2$ be two compact riemannian surfaces isometrically embedded in euclidean spaces, in this paper we will show that if $M = S_1 \times S_2$, then for any function $F : M \to \mathbf{R}$, the graph of F, i.e. the manifold $\{(x, F(x) : x \in M\}$, does not have positive sectional curvature. We also prove a small variation of the isometrically embedded theorem for riemannian manifold in euclidean spaces due to Nash.

§0 Introduction: Let M be a riemannian manifold and let T_pM denoted the tangent vector space of M at p. The sectional curvature is the function that assigns the Gauss curvature at p of the surface built of geodesics starting at p and velocity vector in σ to any 2-dimensional space $\sigma \subset T_pM$. We will say that the riemannian manifold M has positive sectional curvature if for every point $p \in M$ the sectional curvature $K(\sigma)$ of every 2-plane $\sigma \subset T_pM$ is positive. An example of such manifolds are the n dimensional spheres of radius r, $S^n(r)$ with the metric induced by $\mathbf{R}^{n+1}$. In this case its sectional curvature is equal to $\frac{1}{r^2}$ for any 2 plane σ in T_pM. In general the question of deciding if a given manifold admits a riemannian metric with positive sectional curvature is a difficult one. The conjecture stating that no riemannian metric on $S^2 \times S^2$ has positive sectional curvature is known as the Hopf's conjecture.

A direct computation shows that when we endowed $S^2 \times S^2$ with the riemannian metric induced by $\mathbf{R}^6$, then for any $(a, b) \in S^2 \times S^2$ the sectional curvature of a plane spaned by a vector of the form $(v, 0) \in T_{(a,b)}S^2 \times S^2$ and a vector of the form $(0, w) \in T_{(a,b)}(S^2 \times S^2)$, with $v, w \in \mathbf{R}^3$, is zero, we also have that the sectional curvature for the planes $T_{(a,b)}S^2 \times S^2 \cap \mathbf{R}^3 \times \{0\}$ and $T_{(a,b)}S^2 \times S^2 \cap \{0\} \times \mathbf{R}^3$ is 1, for any other plane the sectional curvature is a number between 0 and 1.

In this paper we suggest an idea that may be usefull for trying to prove the Hopf's conjecture. This idea relies on the main 2 theorems proven in this paper, the first one is a small variation of a theorem by Nash that states that any riemannian manifold can be isometrically embedded in some $\mathbf{R}^l$. The second one states that if $M = S_1 \times S_2$ where (S_i, g_i) $i = 1, 2$ are compact riemannian surfaces and M is endowed with the product metric, then the graph of a function $F : M \to \mathbf{R}$ can not have positive sectional curvature.

§1 Preliminaries: Let ∇ be the Levi Civita connection of the riemannian manifold (M, g). Let us denote by $\Gamma(TM)$ the set of tangent vector fields on M. The curvature tensor is defined by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

for any $X, Y, Z \in \Gamma(M)$. Here $[X, Y]$ denotes the Lie bracket of X and Y. Since R is a tensor, then the value of $R(X, Y)Z$ at $m \in M$ only depends on the values of X, Y and Z at $m \in M$, therefore it makes sense to talk about $R(X, Y)Z$ when X, Y, Z are not vector fields but vectors in T_mM. Given a plane $\sigma \in T_mM$, the sectional curvature of σ is defined by $K(\sigma) = g(R(X,Y)X,Y)$ where $\{X,Y\}$ is any orthonormal base of σ with respect to the inner product g. It is not difficult to prove the following:

Remark 1.1: If the manifold M is a subset of $\mathbf{R}^l$ and the metric g is the one induced by $\mathbf{R}^l$ then we can compute the Levi Civita connection of two tangent vector fields X and Y in $\Gamma(TM)$ by

$$(\nabla_X Y)(m) = (J\bar{Y})(m)X(m) = d\bar{Y}(X(m))$$

where $\bar{Y} : U \subset \mathbf{R}^l \to \mathbf{R}^l$ is any vector field defined in a neighborhood U of m in $\mathbf{R}^l$ such that $\bar{Y}(m) = Y(m)$ for any $m \in M$, and $J\bar{Y}$ is the jacobian matrix or the matrix of the partial derivatives. In the comment above we are viewing T_mM as a subset of $\mathbf{R}^l$, henceforth a tangent vector field $Z \in \Gamma(MT)$ is a map from M to $\mathbf{R}^l$ with the property that $Z(m) \in T_mM$.

§2 Main Theorems: Let $M \subset \mathbf{R}^N$ be an embedded smooth compact n dimensional manifold and let g_0 be the riemannian metric induced by $\mathbf{R}^N$. In this paper we prove the following theorems:

Theorem A: *Let M be a n-dimensional manifold embedded in $\mathbf{R}^N$. For any riemannian metric g on M there exists a positive real constant C, a positive integer k and a smooth function $F : M \to \mathbf{R}^k$ such that the manifold $\{(x, F(x)) \in \mathbf{R}^{N+k} : x \in M\}$ with the metric induced by $\mathbf{R}^{N+k}$ is isometric to the riemannian manifold (M, Cg)*

Theorem B: *Let $S_i \subset \mathbf{R}^{k_i}$, $i = 1, 2$ be two compact riemannian surfaces with the metric induced by the euclidean spaces. If $M = S_1 \times S_2$, in particular $k_1 + k_2 = N$, then for any smooth function $F : M \to \mathbf{R}$ the manifold $\{(x, F(x)) \in \mathbf{R}^{N+1} : x \in M\}$ with the metric induced by $\mathbf{R}^{N+1}$ does not have positive sectional curvature.*

Proof of theorem A: By definition of riemannian metric we have that for any $m \in M$ there exists a constant $C(m)$ such that for any pair of vectors v and w in T_mM

$$g_0(v,w) < C(m)g(v,w) \text{ for any pair of vectors } v \text{ and } w \text{ in } T_mM$$

Since M is compact we can find a constant C such that the inequality above hold for every $m \in M$, this can be done by diagonalizing the metric g_0 with respect to the metric g and considering the function that assigns the greatest eigenvalue in this diagonalization to any $m \in M$; this function is continuous and we can take C as any number bigger than the maximum of this function over M.

By the definition of C we have that the 2-tensor given by $\hat{g} = Cg - g_0$ is symmetric and positive defined, therefore it defines a riemannian metric on M. By Nash's theorem [N] or [KN, p. 354] we have that there exists an imbedding $F : M \to \mathbf{R}^k$ such that the pull back of the metric induced by $\mathbf{R}^k$ is the metric $\hat{g}$, i.e. for any $m \in M$

$$\langle dF_m(v), dF_m(w)\rangle = Cg(v,w) - g_0(v,w) \text{ for any pair of vectors } v \text{ and } w \text{ in } T_mM$$

Here $\langle\ ,\ \rangle$ denotes the inner product in euclidean spaces. It is clear that this function F satisfies the conditions of Theorem A because the metric induced on M by the imbedding $\{(x,F(x)) \in \mathbf{R}^{N+1} : x \in M\}$ is the one induced by the map $\phi(x) = (x,F(x))$ for all $x \in M$, and for any pair of vectors v,w in T_mM we have

$$\langle d\phi_m(v), d\phi(w)\rangle = \langle (v, dF_m(v)), (w, dF_m(w))\rangle = g_0(v,w) + \langle dF_m(v), dF_m(w)\rangle = Cg(v,w)$$

Notice that the inner product in $\mathbf{R}^N$ restricted to T_mM is given by g_0. ■

The theorem above tells us that given a riemannian metric g_0 on M, any other riemannian metric on M can be realized, up to a constant, as the natural metric on the graph of a function from M to an euclidean space.

Proof of Theorem B: Let us denote by $\bar\nabla$ the connection on M with the metric induced by the embedding $\phi(x) = (x,F(x))$ of M in $\mathbf{R}^{N+1}$ and let us denote by ∇ the connection on M induced by $\mathbf{R}^N$. We will use the following notation.

1. If $m \in M$, we will denote by $\bar m$ the point $(m, F(m))$

2. If a map $Y : M \to \mathbf{R}^N$ defines a tangent vector field on M, we denote by $\bar Y$ the vector field on $\bar M$ defined by $\bar Y(\bar m) = (Y(m), dF_m(Y(m)))$.

3. Given $p \in M$ and $v \in \mathbf{R}^N$, we will denote by $v^T(p)$ the tangent orthogonal projection of v on T_pM. Given $w \in \mathbf{R}^{N+1}$ we will denote by $w^{\bar T}(\bar p)$ the tangent orthogonal projection of w on $T_{\bar p}\bar M$.

From now on, we will refer to the riemannian manifold M with the metric g_0 just as the manifold M, and we will refer to the manifold M with the metric induced by the embedding $\phi(p) = (p, F(p))$ just as the manifold $\bar M$. We will find the sectional curvature on $\bar M$ in terms of the sectional curvature of M and the derivatives of F. For any $m \in M \subset \mathbf{R}^N$ let $\{v_i : i = 1, \dots n\}$ be an orthonormal frame defined in a open neighborhood $U \subset M$ of m, notice that each $v_i : U \to \mathbf{R}^N$ is a tangent vector field. Without loss of generality we may assume that the vector fields $\nabla_{v_i} v_j$ vanish at m for all i,j. We will denote by ∇F the gradient vector of F as a function on M; since the frame of the vector field $v_i^\prime s$ is orthonormal then for any $p \in U$ we have that $\nabla F = \sum_{i=1}^n dF_p(v_i(p))v_i(p)$. Recall that the hessian of F is the symmetric 2-tensor given by $\mathrm{Hess}(F)(X,Y) = \langle \nabla_X(\nabla F), Y\rangle$ for any pair of tangent vector fields on M. For any $p \in U$ we will define $F_i(p) = dF_p(v_i(p))$ and $F_{ij}(p) = (\mathrm{Hess}(F))_p(v_i(p), v_j(p))$. Before trying to find a relation between the sectional curvature of M and $\bar M$ we need to prove the following lemmas,

Lemma 2.1: **(a)** *The inverse of the matrix* $\{g_{ij}\}_{i,j=1}^n$ *defined by* $g_{ij} = \delta_{ij} + F_iF_j$ *is the matrix* $\{g^{ij}\}_{i,j=1}^n$ *defined by*

$$g^{ij} = \delta_{ij} - \frac{F_i F_j}{1+|\nabla F|^2}$$

(b) *If v is a vector in $\mathbf{R}^N$ and r is a real number then for any $m \in M$ we have that*

$$(v,r)^{\bar{T}} = \big(v^T, \langle v, \nabla F\rangle\big) + \frac{r - \langle v, \nabla F\rangle}{1+|\nabla F|^2}\big(\nabla F, |\nabla F|^2\big) = \overline{v^T + \frac{r - \langle v, \nabla F\rangle}{1+|\nabla F|^2}\nabla F}$$

Proof of the lemma: A direct computation shows that $\sum_{j=1}^n g_{ij} g^{jk} = \delta_{ik}$ therefore **(a)** follows. Let us prove **(b)**. For any $m \in M$ let us define $\{v_i : i = 1, \dots, n\}$ as above, then we have that the vectors $\{w_i = \bar{v}_i(\bar{m}) : i = 1, \dots, n\}$ form a base for $T_{\bar{m}}\bar{M}$, therefore, there exist numbers $c_1, \dots, c_n$ such that

$$(v,r)^{\bar{T}}(\bar{m}) = \sum_{i=1}^n c_i w_i \tag{1}$$

We have that $\langle w_i, w_j\rangle = \langle (v_i, dF_p(v_i)), (v_j, dF_p(v_j))\rangle = \delta_{ij} + F_i(m)F_j(m) = g_{ij}$, if we multiply the equation (1) by w_j we obtain that $\langle v, v_j\rangle + rF_j = \sum_{i=1}^n c_i g_{ij}$, now if we multiply this equation by g^{jk} and we add from $j = 1$ to $j = n$ we obtain that

$$\begin{aligned} c_k = \sum_{j=1}^n (\langle v, v_j\rangle + rF_j) g^{jk} =& \langle v, v_k\rangle + rF_k - \sum_{j=1}^n \{\frac{\langle v, v_j\rangle F_j F_k}{1+|\nabla F|^2} + \frac{rF_j F_j F_k}{1+|\nabla F|^2}\} \\ =& \langle v, v_k\rangle + rF_k - \frac{\langle v, \nabla F\rangle F_k}{1+|\nabla F|^2} - \frac{r|\nabla F|^2 F_k}{1+|\nabla F|^2} \end{aligned}$$

Plugging these values for c_k in equation (1) we obtain that

$$\begin{aligned} (v,r)^{\bar{T}}(\bar{m}) =& \sum_{k=1}^n \big(\langle v, v_k\rangle + rF_k - \frac{\langle v, \nabla F\rangle F_k}{1+|\nabla F|^2} - \frac{r|\nabla F|^2 F_k}{1+|\nabla F|^2}\big)(v_k, F_k) \\ =& \big(v^T, \langle v, \nabla F\rangle\big) + r\big(\nabla F, |\nabla F|^2\big) - \frac{\langle v, \nabla F\rangle + r|\nabla F|^2}{1+|\nabla F|^2}\big(\nabla F, |\nabla F|^2\big) \\ =& \big(v^T, \langle v, \nabla F\rangle\big) + \frac{r - \langle v, \nabla F\rangle}{1+|\nabla F|^2}\big(\nabla F, |\nabla F|^2\big) \end{aligned}$$

This proves the lemma.

Lemma 2.2: *If $v, w : M \to \mathbf{R}^N$ are tangent vector fields on M, then the Levi Civita connection of the tangent vector fields $\bar{v}, \bar{w} : \bar{M} \to \mathbf{R}^{N+1}$ is given by*

$$\bar{\nabla}_{\bar{v}}\bar{w} = \overline{\nabla_v w + \frac{\mathrm{Hess}F(v,w)}{1+|\nabla F|^2}\nabla F}$$

Proof of the lemma: Let $\alpha(t)$ be a smooth curve on $M \subset \mathbf{R}^N$ such that $\alpha(0) = m$ and $\alpha'(0) = v$ and define $\beta(t) = (\alpha(t), F(\alpha(t)))$. Since the riemannian metrics on M and $\bar{M}$ are those induced by the euclidean spaces, then we have that

$$
\begin{aligned}
(\bar{\nabla}_{\bar{v}}\bar{w})(\bar{m}) =& \Big(\frac{d}{dt}\bar{w}(\beta(t))\Big)^{\bar{T}} \\
=& \Big(\frac{d}{dt}w(\alpha(t)), \frac{d}{dt}\langle \nabla F, w\rangle(\alpha(t))\Big)^{\bar{T}} \\
=& \Big(\frac{d}{dt}w(\alpha(t)), \langle \nabla_v \nabla F, w\rangle + \langle \nabla F, \nabla_v w\rangle\Big)^{\bar{T}}
\end{aligned}
$$

Using part (b) of Lemma 2.1 and the definition of Hess(F) we conclude the lemma.

Since Hess(F) is symmetric, we have as a corollary of the lemma above that if $[v, w]$ vanishes at $m \in M$, then $[\bar{v}, \bar{w}]$ also vanishes at $\bar{m} \in M$.

Let us denote the covariant derivative of the tensor Hess(F) by D^2dF, i.e. for any tangent vector fields X, Y and Z we have

$$
D^2dF(X, Y; Z) = Z(\text{Hess}(F)(X, Y)) - \text{Hess}(F)(\nabla_Z X, Y) - \text{Hess}(F)(X, \nabla_Z Y)
$$

Lemma 2.3: *Let R and $\bar{R}$ denote the curvature tensor of M and $\bar{M}$ respectively. For any tangent vector fields $X, Y, Z : M \to \mathbf{R}^N$ in M we have that*

$$
\begin{aligned}
\bar{R}(\bar{X}, \bar{Y})\bar{Z} =& \overline{R(X, Y)Z} + \frac{\text{Hess}(F)(X, Z)}{1 + |\nabla F|^2}\overline{\nabla_Y \nabla F} - \frac{\text{Hess}(F)(Y, Z)}{1 + |\nabla F|^2}\overline{\nabla_X \nabla F} \\
&+ \frac{D^2dF(X, Z; Y) - D^2dF(Y, Z; X)}{1 + |\nabla F|^2}\overline{\nabla F} \\
&+ \frac{\text{Hess}(F)(X, Z)\text{Hess}(F)(Y, \nabla F) - \text{Hess}(F)(Y, Z)\text{Hess}(F)(X, \nabla F)}{1 + |\nabla F|^2}\overline{\nabla F}
\end{aligned} \tag{2}
$$

Proof of Lemma 2.3 Given $m \in M$, let us define $\{v_i : i = 1, \dots n\}$ an orthonormal frame like above. In order to prove the lemma, it is enough to prove the equation (2) at the point m in the case $X = v_i$, $Y = v_j$ and $Z = v_k$. Using the definition of curvature tensor and the lemma 2.2 we have that,

$$
\begin{aligned}
\bar{R}(\bar{v_i},\bar{v_j})\bar{v_k} =&\bar{\nabla}_{\bar{v_j}}\bar{\nabla}_{\bar{v_i}}\bar{v_k} - \bar{\nabla}_{\bar{v_i}}\bar{\nabla}_{\bar{v_j}}\bar{v_k} \\
=&\bar{\nabla}_{\bar{v_j}}(\overline{\nabla_{v_i}v_k} + \frac{F_{ik}}{1+|\nabla F|^2}\overline{\nabla F}) - \bar{\nabla}_{\bar{v_i}}(\overline{\nabla_{v_j}v_k} + \frac{F_{jk}}{1+|\nabla F|^2}\overline{\nabla F}) \\
=&\overline{\nabla_{v_j}\nabla_{v_i}v_k} + \overline{\nabla_{v_j}\big(\frac{F_{ik}}{1+|\nabla F|^2}\nabla F\big)} + \mathrm{Hess}(F)\big(v_j, \frac{F_{ik}}{1+|\nabla F|^2}\nabla F\big)\overline{\nabla F} \\
&- \overline{\nabla_{v_i}\nabla_{v_j}v_k} - \overline{\nabla_{v_i}\big(\frac{F_{jk}}{1+|\nabla F|^2}\nabla F\big)} - \mathrm{Hess}(F)\big(v_i, \frac{F_{jk}}{1+|\nabla F|^2}\nabla F\big)\overline{\nabla F} \\
=&\overline{R(v_i,v_j)v_k} + \frac{F_{ik}}{1+|\nabla F|^2}\overline{\nabla_{v_j}\nabla F} - \frac{F_{jk}}{1+|\nabla F|^2}\overline{\nabla_{v_i}\nabla F} \\
&+ \big\{\frac{F_{ik}\mathrm{Hess}(F)(v_j,\nabla F) - F_{jk}\mathrm{Hess}(F)(v_i,\nabla F)}{1+|\nabla F|^2}\big\}\overline{\nabla F} \\
&+ \big\{v_j\big(\frac{F_{ik}}{1+|\nabla F|^2}\big) - v_i\big(\frac{F_{jk}}{1+|\nabla F|^2}\big)\big\}\overline{\nabla F}
\end{aligned}
$$

Using that $\nabla_{v_j}v_i(m)$ and $[v_i,v_j](m)$ vanish for any $i,j\in\{1,\dots,n\}$ we can prove that

$$
\begin{aligned}
v_j\big(\frac{F_{ik}}{1+|\nabla F|^2}\big) - v_i\big(\frac{F_{jk}}{1+|\nabla F|^2}\big) =&\frac{v_j(F_{ik}) - v_i(F_{jk})}{1+|\nabla F|^2}(m) + [v_j,v_i]((1+|\nabla F|^{-1}) \\
=&\frac{D^2df(v_i,v_k;v_j) - D^2dF(v_j,v_k;v_i)}{+|\nabla F|^2}(m)
\end{aligned}
$$

This equality together with the expression for $\bar{R}(\bar{v_i},\bar{v_j})\bar{v_k}$ that we obtain above give us the proof of the lemma.

Now we are ready to prove theorem B. By Morse theory [B], we have that there exists a closed geodesic $\gamma_1\subset S_1$ and a closed geodesic $\gamma_2\subset S_2$. Let us define $T=\{(x,y)\in M : x\in\gamma_1 \text{ and } y\in\gamma_2\}$. Let $h:T\to\mathbf{R}$ be the function defined by $h(m)=F(m)$ for all $m\in T$. For every $z\in S_i$, let $\psi_i:\gamma_i\to\mathbf{R}^{k_i}$ be unit tangent vector fields, $i=1,2$. Let $\phi_i:T\to\mathbf{R}^N$ be the tangent vector field defined by $\phi_1(x,y)=(\psi_1(x),0,\dots,0)$ and $\phi_2(x,y)=(0,\dots,0,\psi_2(y))$. We will first prove the theorem in the case that h is a Morse function on T. Let $(x_0,y_0)\in T$ be a critical point of h which is a saddle point, i.e. the determinant of Hess(h) at (x_0,y_0) is negative, this saddle point exists because T is topologically a torus. For any $(x,y)\in T$ we can write $\nabla F(x,y)=\nabla h+(\nabla F)^\perp$ where $(\nabla F)^\perp$ is perpendicular to any vector in $T_{(x,y)}T$. It is not difficult to show that T is a totally geodesic submanifold of M [D], therefore we have that

$$
\begin{aligned}
\mathrm{Hess}(F)(\phi_i,\phi_j) =&\langle\nabla_{\phi_i}\nabla F,\phi_j\rangle \\
=&\langle\nabla_{\phi_i}\nabla h,\phi_j\rangle + \langle\nabla_{\phi_i}(\nabla F)^\perp,\phi_j\rangle \\
=&\langle D_{\phi_i}\nabla h,\phi_j\rangle - \langle(\nabla F)^\perp,\nabla_{\phi_i}\phi_j\rangle \\
=&\mathrm{Hess}(h)(\phi_i,\phi_j)
\end{aligned}
\tag{4}
$$

In the last equality above we have used the fact that since T is totally geodesic, then

$\nabla_{\phi_i}\phi_j(x,y) \in T_{(x,y)}T$ for every $(x,y) \in T$; the D in the expression above denotes the Levi Civita connection on T.

Recall that $\langle \bar{X}, \bar{Y}\rangle = \langle X, Y\rangle + dF(X)dF(Y)$. We will prove the theorem by showing that $\langle \bar{R}(\bar{\phi_1}, \bar{\phi_2})\bar{\phi_1}, \bar{\phi_2}\rangle$ at $(x_0, y_0, F(x_0, y_0)$ is negative. Since (x_0, y_0) is a critical point in M then $dh_{(x_0,y_0)}(\phi_i) = dF_{(x_0,y_0)}(\phi_i) = \langle \nabla F, \phi_j\rangle(x_0, y_0)$ vanishes. Using the lemma 2.3 and the fact that the sectional curvature of the plane spanned by $\{\phi_1, \phi_2\}$ is zero because M has the product metric [D], we obtain at the point (x_0, y_0) that

$$\begin{aligned}
\bar{R}(\bar{\phi_1}, \bar{\phi_2})\bar{\phi_1}, \bar{\phi_2}\rangle =& \langle R(\phi_1), \phi_2)\phi_1, \phi_2)\\
&+\frac{\text{Hess}(F)(\phi_1,\phi_1)}{1+|\nabla F|^2}\langle \nabla_{\phi_2}\nabla F, \phi_2\rangle - \frac{\text{Hess}(F)(\phi_2,\phi_1)}{1+|\nabla F|^2}\langle \nabla_{\phi_1}\nabla F, \phi_2\rangle\\
&+\frac{D^2dF(\phi_1,\phi_1;\phi_2)}{1+|\nabla F|^2}\langle \nabla F, \phi_2\rangle - \frac{D^2dF(\phi_2,\phi_1;\phi_1)}{|\nabla F|^2}\langle \nabla F, \phi_2\rangle\\
&+\frac{\text{Hess}(F)(\phi_1,\phi_1)\text{Hess}(F)(\phi_2,\nabla F)}{1+|\nabla F|^2}\langle \nabla F, \phi_2\rangle\\
&-\frac{\text{Hess}(F)(\phi_2,\phi_1)\text{Hess}(F)(\phi_1,\nabla F)}{1+|\nabla F|^2}\langle \nabla F, \phi_2\rangle\\
=&\frac{\text{Hess}(h)(\phi_1,\phi_1)Hess(h)(\phi_2,\phi_2)}{1+|\nabla F|^2} - \frac{\text{Hess}(h)(\phi_2,\phi_1)Hess(h)(\phi_1,\phi_2)}{1+|\nabla F|^2}\\
<& \quad 0
\end{aligned}$$

This inequality completes the proof in the case that the function $h : T \to \mathbf{R}$ is a Morse function. For the general case we will proceed by contraction. Let us assume that M has positive sectional curvature. Since M is compact, then there exists $\epsilon > 0$ such that for any σ in T_pM, we have that $K(\sigma) > \epsilon$. Let T be defined as above. Since M is compact we can use the formula in lemma 2.3 for the curvature tensor in order to find a positive δ such that if $\bar{F} : M \to \mathbf{R}$ is chosen such that the difference between $\bar{F}$ and its derivatives up to third order with F and its derivatives up to third order respectively is less than δ, then the difference between the sectional curvature of the graph of F and the graph of $\bar{F}$ is less than ϵ. By Morse Theory [M] we can choose this $\bar{F}$ such that the function $\bar{F}$ restricted to T is a Morse function. Using the case we already considered we have that the graph of $\bar{F}$ has a plane with negative sectional curvature. This is a contradiction with the fact that the difference between the sectional curvature of the graph of F and the graph of $\bar{F}$ is less than ϵ. ■

Remark 2.1: As a consequence of Theorem A and the fact that the sign of the sectional curvature does not change when we multiply a riemannian metric by a constant, we have that if we succeed showing that for any $F : S^2 \times S^2 \to \mathbf{R}^k$ the manifold $\{(x, F(x)) \in \mathbf{R}^6 + k : x \in S^2 \times S^2\}$ with the metric induced by $\mathbf{R}^{6+k}$ has not positive sectional curvature, then the Hopf's conjecture would be true. Notice that Theorem B provides a proof in the case $k = 1$.

Bibliography

[B] Bott, R. *Lectures on Morse theory, old and new*, Bull. AMS **Vol 7**, Number 2, (1982), pp 331-358.

[D] Do Carmo, M. *Riemannian Geometry*, Birkhauser, 1992.

[KN] Kobayashi, Nomizu *Foundation of differential geometry* **Vol I-II**, John Wiley & Sons, Inc. 1996.

[M] Milnor, J. *Morse Theory* Princeton University Press, 1963.

[N] Nash, J. F. *The imbedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956), pp 20-63.