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A characteristic property of spherical caps

Jaime Ripoll

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Abstract It is proved that given $H \geq 0$ and an embedded compact orientable constant mean curvature H surface M included in the half space $z \geq 0$, not everywhere tangent to $z = 0$ along its boundary $\partial M = \gamma \subset \{z = 0\}$, the inequality

$$H \leq (\min \kappa)\left(\min \sqrt{1 - (\kappa_g/\kappa)^2}\right)$$

is satisfied, where κ and κ_g are the geodesic curvatures of γ on $z = 0$ and on the surface M, respectively, if and only if M is a spherical cap or the planar domain enclosed by γ. The equivalence is no longer true if M is assumed to be only complete.

Keywords Spherical cap · Constant mean curvature

Mathematics Subject Classification (2000) 53A10 · 53C42

1 Introduction

In this note we prove:

Theorem *Let γ be a smooth curve in the plane $z = 0$ in $\mathbb{R}^3$ and M an orientable, compact embedded surface with constant mean curvature $H \geq 0$ included in the half space $z \geq 0$ with boundary $\partial M = \gamma$ and not everywhere tangent to $z = 0$ along γ. Let κ be the planar*

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J. Ripoll (✉)
Instituto de Matemática, Universidade Federal do R. G. do Sul, Av. Bento Gonçalves 9500,
91501-570 Porto Alegre, RS, Brazil
e-mail: ripoll@mat.ufrgs.br

curvature of γ, and κ_g be the geodesic curvature of γ in M. Then

$$H \leq (\min \kappa) \left(\min \sqrt{1 - \left(\frac{\kappa_g}{\kappa} \right)^2} \right) \tag{1}$$

if and only if M is a spherical cap or the domain enclosed by γ.

Observe that if M is a constant mean curvature (CMC) H half cylinder orthogonal to $z = 0$ then

$$H = \frac{\kappa}{2} = \frac{\kappa}{2} \sqrt{1 - \left(\frac{\kappa_g}{\kappa} \right)^2} < (\min \kappa) \left(\min \sqrt{1 - \left(\frac{\kappa_g}{\kappa} \right)^2} \right)$$

which shows that the result is false if M is not assumed to be compact.

The above theorem relates to a conjecture of Brito et al. [2] asserting that if M is an embedded compact CMC surface M in a half space $z \geq 0$ of $\mathbb{R}^3$ whose boundary is a convex curve on the plane $z = 0$ then M has genus zero. In [5], Ros and Rosenberg prove that under the assumptions of the conjecture, there is an upper bound $C(\min \kappa, \max \kappa)$ for the mean curvature H of M, depending only on the extremum values $\min \kappa$ and $\max \kappa$ of the curvature κ of γ which implies the conjecture. As far as we know, there are not an explicit formula nor estimates of $C(\min \kappa, \max \kappa)$. If γ is a geodesic in M then our result implies that one can take $C(\min \kappa, \max \kappa) = \min \kappa$.

Proof of the theorem Let M as in the hypothesis of the theorem. Orient M by a normal vector N such that $\langle N, \nu \rangle \geq 0$ along γ, where ν is the unit vector field orthogonal to γ in π pointing to the region enclosed by γ. Let H be the mean curvature of M with respect to N. It is a consequence of the maximum principle (or the tangency principle for CMC surfaces) that with this choice of N we have $H \geq 0$.

We assume that condition (1) is satisfied and prove that M is a spherical cap or the domain in $z = 0$ enclosed by γ. The other implication is elementary.

If $H = 0$ then M is obviously the domain enclosed by γ in $z = 0$ and the theorem is proved in this case. Thus, let us assume that $H > 0$. From (1) it follows that $\min \kappa > 0$. The main idea of the proof is to "close" M along γ with a CMC H surface G ($\partial G = \gamma$) such that $M \cup G$ is an embedded compact topological surface in $\mathbb{R}^3$; moreover, G is constructed such that the tangent planes of G and M at a common point of γ form an interior angle smaller than or equal to π. This allows the application of the Alexandrov reflection technique (see [1]) to prove that M is a graph or that $M \cup G$ has a horizontal symmetry which, as we will see, implies that M is a spherical cap.

We first consider the case that $\min \kappa > H$. Let $\Omega \subset z = 0$ be the domain enclosed by γ ($\partial \Omega = \gamma$). We prove the existence of a CMC H graph G^* of a function $u \in C^2(\overline{\Omega})$ contained in the half-space $z \geq 0$ with $\partial G^* = \gamma$. The graph of $u \in C^2(\Omega)$ is a surface with CMC H with respect to a normal field N such that $\langle N, e_3 \rangle \leq 0$, $e_3 = (0, 0, 1)$, if and only if u satisfies the PDE

$$Q_H(u) := \operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} + 2H = 0. \tag{2}$$

Given $p \in \partial \Omega$, since from (1)

$$\kappa(p) \geq \min \kappa > H,$$

there is a circle C_p of radius $1/\min\kappa$ through p, contained in $z = 0$ and enclosing Ω. Moreover, C_p is the boundary of a CMC H spherical cap which, over Ω, is given as a graph of a nonnegative C^2 function v_p in Ω such that

$$\max|v_p| \leq 1/H, \quad |\nabla v_p(p)| = \frac{H}{\sqrt{\min\kappa^2 - H^2}}. \tag{3}$$

Since $v \equiv 0$ is a subsolution ($Q_H(v) \geq 0$) and v_p a supersolution ($Q_H(v_p) = 0 \leq 0$), $p \in \partial\Omega$, for (2), it follows from the Comparison Principle (Theorem 10.1 of [4]) that if $u \in C^2(\overline{\Omega})$ is a solution of (2) such that $u|_{\partial\Omega} = 0$ then $0 \leq u \leq v_p$, for any $p \in \partial\Omega$. Hence, from (3)

$$\max|u| \leq \frac{1}{H}, \quad \max_{\partial\Omega}|\nabla u| < \frac{H}{\sqrt{\min\kappa^2 - H^2}}.$$

By the Gradient Maximum Principle (Theorem 15.1 of [4]) we have that the norm of the gradient of u is bounded (strictly) in $\overline{\Omega}$ by $H/\sqrt{\min\kappa^2 - H^2}$. We then have the a priori C^1 estimates of a solution $u \in C^2(\overline{\Omega})$ of (2)

$$\max|u| \leq \frac{1}{H}, \quad \max_{\Omega}|\nabla u| < \frac{H}{\sqrt{\min\kappa^2 - H^2}}.$$

It follows from the Continuity Method (Theorem 13.8 of [4]) the existence of a solution $u \in C^2(\overline{\Omega})$ of (2) such that $u \geq 0$ and $u|_{\partial\Omega} = 0$. Let G^* be the graph of u and denote by G the reflection of G^* on the plane $z = 0$. Then $G \subset \{z \leq 0\}$ and $B := M \cup G$ is an embedded, compact topological surface without boundary which is a C^∞ differentiable manifold with CMC H in $B\backslash\partial\Omega$. We prove that the tangent planes of M and G at a common point of γ form an inner angle which is smaller than π. Denote by N_1 the unit normal vector of M pointing to the bounded component of $\mathbb{R}^3\backslash B$, and by N_2 the one of G. Observe then that the assertion is trivially true at the points of γ where $\langle N_1, e_3\rangle < 0$. If $\langle N_1, e_3\rangle \geq 0$ we then have to prove that

$$\langle N_1(p), n(p)\rangle > \langle N_2(p), n(p)\rangle, \qquad p \in \gamma, \tag{4}$$

where n is the unit normal vector of γ in the plane $z = 0$, pointing to Ω.

Considering a parametrization of γ by arc length, the plane curvature of γ is $\kappa = |\gamma''|$ and the geodesic curvature of γ in M is $\kappa_g = \left|\langle\gamma'', \eta\rangle\right|$, where η is the unit vector field tangent to M and normal to γ'. Note that $\kappa > 0$ since $\min\kappa > H > 0$. Then

$$n = \gamma''/|\gamma''| = \gamma''/\kappa$$

and, since

$$\gamma'' = \langle N_1, \gamma''\rangle N_1 + \langle\eta, \gamma''\rangle\eta$$

we obtain

$$\langle N_1, \gamma''\rangle^2 = |\gamma''|^2 - \langle\eta, \gamma''\rangle^2 = \kappa^2 - \kappa_g^2$$

so that

$$\langle N_1, n\rangle^2 = \frac{\langle N_1, \gamma''\rangle^2}{\kappa^2} = 1 - \frac{\kappa_g^2}{\kappa^2} \geq \min\left(1 - \left(\frac{\kappa_g}{\kappa}\right)^2\right). \tag{5}$$

Since, from (3),

$$\langle N_2, n\rangle^2 < \frac{H^2}{(\min\kappa)^2}$$

we see that inequality (4) is implied by the hypothesis. Using now the Alexandrov technique (see [1]) by considering reflections of B on horizontal planes (that is, planes $z = c$) coming from $z = +\infty$, and observing that the reflected part of B which is on the interior of B can never touch γ (since the tangent planes to M and to G along γ have an inner angle strictly smaller than π), we easily conclude that either B has a horizontal symmetry plane or M is a graph. In the first case it follows that B is differentiable compact manifold embedded in $\mathbb{R}^3$ with CMC H so that, by Alexandrov's theorem, B is a sphere. We have therefore proved that if $\min\kappa > H$ then M is a graph or a spherical cap.

Consider now the case that $\min\kappa = H > 0$. It follows from (1) that in this situation $\kappa_g = 0$, that is, γ is a geodesic on M and it follows from (5) that M is orthogonal to $z = 0$ along γ. By the Continuation Principle the reflection M^* of M on the plane $z = 0$ is a CMC H surface and $M \cup M^*$ is an embedded compact CMC H surface without boundary. By Alexandrov's theorem [1], $M \cup M^*$ is a sphere. It follows that M is a spherical cap.

We have therefore proved that in any case condition (1) implies that M is a spherical cap or a graph. But we will now see that if a graph satisfies condition (1) then it is a spherical cap (hence, a small spherical cap). Being M a graph given by a function u, we have

$$N_1 = \nabla u - \frac{1}{\sqrt{1+|\nabla u|^2}} e_3$$

and

$$|\nabla u| n = \nabla u$$

so that

$$\langle N_1, n\rangle = \frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}}. \tag{6}$$

Since from (1) $\kappa_{\min} \geq H$, we may apply again the Gradient Maximum Principle to obtain the estimate

$$\sup|\nabla u| \leq \frac{H}{\sqrt{\min\kappa^2 - H^2}}. \tag{7}$$

Since condition (1) is equivalent to

$$\frac{H^2}{\min\kappa^2} \leq 1 - \frac{\max\kappa_g^2}{\min\kappa^2}, \tag{8}$$

we have from (6) and (7)

$$\frac{H^2}{\min\kappa^2} \geq \frac{|\nabla u|^2}{1+|\nabla u|^2} = \langle N_1, n\rangle^2 \geq 1 - \frac{(\max\kappa_g)^2}{(\min\kappa)^2}$$

so that, from (8),

$$\frac{|\nabla u|^2}{1+|\nabla u|^2} = 1 - \frac{(\max\kappa_g)^2}{(\min\kappa)^2} = \text{const}$$

and this implies that $|\nabla u|$ is constant along γ.

We shall now apply the Alexandrov reflection technique to prove that M is a spherical cap. Since it is by now a well known technique, we just observe that when considering reflections of M on a family of parallel planes orthogonal to $z = 0$, the constancy of $|\nabla u|$ along γ implies that the intersection between the non reflected and reflected parts M^- and M^* of M, respectively, at the last point where the reflected part M^* is still interior of M, is a boundary tangency between M^- and M^*. Therefore, the Boundary Tangency Principle implies that $M^* = M^-$. Since this symmetry occurs on any direction of the plane $z = 0$ it follows that M is invariant by a rotational group of isometries, that is, M is a surface of revolution. From the classical work of Delaunay [3] on rotational CMC surfaces it follows that M is a small spherical cap. This concludes with the proof of the theorem.

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