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A Rigidity Theorem for Compact Hypersurfaces with an Upper Bound for the Ricci Curvature

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Abstract. We focus our attention on compact hypersurfaces with Ricci curvature bounded from above and we give a sufficient condition for them to be spherical. This generalizes and completes previous results.

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1. Introduction and the Main Results

The classical Bonnet–Myers [5] theorem asserts that if the Ricci curvature of a complete Riemannian manifold M_n is greater than or equal to $(n-1)c$ with $c > 0$, then the diameter of M_n is less than or equal to $\pi/\sqrt{c}$ and so M_n is compact. If in addition, $\mathrm{diam}(M_n) = \pi/\sqrt{c}$, then M_n is isometric to the sphere $S^n(c)$ of constant curvature c (Cheng maximal diameter sphere theorem). These are some of the most famous curvature-topology results and illustrate what some call '*la domination universelle de la courbure de Ricci*' (M. Berger quoted in [5]). In comparison, there are not too many results about manifolds with Ricci curvature bounded from above. For compact hypersurfaces, this work will show a maximal theorem, i.e.: with an additional assumption, such manifolds are isometric to spheres of constant curvature.

More precisely, let $\tilde{S}_{n+1}(c)$ be the simply connected space form of constant curvature $c (c \in \mathbb{R})$ and $S_p(r) = \{q \in \tilde{S}_{n+1}(c)/d_p(q) = r\}$ the distance sphere with center $p \in \tilde{S}_{n+1}(c)$ and radius $r > 0$ where d is the Riemannian distance of $\tilde{S}_{n+1}(c)$. If $c > 0$, assume that $r < \pi/(2\sqrt{c})$. It is a classical result [1] that $S_p(r)$ is a compact connected umbilical hypersurface with principal curvatures equal to

$$k_c(r) = \begin{cases} \sqrt{c}\,\cot(r\sqrt{c}) & \text{if } c > 0 \\ 1/r & \text{if } c = 0, \\ \sqrt{-c}\,\coth(r\sqrt{-c}) & \text{if } c < 0. \end{cases}$$

The Ricci curvature of $S_p(r)$ is therefore constant and equal to

$$\mathrm{Ric} = (n-1)(c + k_c^2(r)).$$

Conversely, we prove the following result:

THEOREM 1. *Let N be a compact connected hypersurface of the space form $\tilde{S}_{n+1}(c)$ ($n \geqslant 3$) included in a closed normal ball $\overline{B}_p(r)$ of M with $r < \pi/(2\sqrt{c})$ if $c > 0$ and assume that the Ricci curvature of N satisfies*

$$\mathrm{Ric} \leqslant (n-1)(c + k_c^2(r)).$$

Then $N = \partial \overline{B}_p(r) = S_p(r)$.

This generalizes the previous results of Deshmukh [3] and Leung [8]. Moreover, for $n = 2$ and $c = 0$, Theorem 1 is also true and was proved by Koutroufiotis [7]. Theorem 1 is an immediate corollary of the following theorem:

THEOREM 2. *Let M_{n+1} be a complete Riemannian manifold ($n \geqslant 3$) with sectional curvature K^M bounded from above by a constant c. Let $i: N \to M$ be an isometric embedding of a n-dimensional compact connected Riemannian manifold with $i(N)$ included in a closed normal ball $\overline{B}_p(r)$ of M with $r < \pi/(2\sqrt{c})$ if $c > 0$. Assume that the Ricci curvature of N satisfies*

$$\mathrm{Ric} \leqslant \mathrm{Ric}^M - K^M(\cdot, \nu) + (n-1)k_c^2(r),$$

where ν is a normal field to $i(N)$ in M. Then $i(N)$ is the geodesic hypersphere $\partial\overline{B}_p(r) = S_p(r)$ and N is isometric to the sphere $(S^n, g_c(r))$ with its canonical metric $g_c(r)$ of constant curvature $K = k_c^2(r) + c$. Moreover, the interior of $(\overline{B}_p(r))$ is isometric to the ball of radius r in the space form $\tilde{S}_{n+1}(c)$ of constant curvature c.

A similar result was obtained by Markvorsen in [9] by an estimate on the mean curvature of N.

2. Proof of Theorem 2

In the sequel, the notations of the theorem will be used. Denote also $i(N) = \tilde{N}$.

Step 1. As $\tilde{N}$ is included in a closed normal ball $\overline{B}_p(r)$, the function $f = \frac{1}{2}d_p^2: \tilde{N} \to \mathbb{R}$ is smooth. The Hessian of f, for the induced Riemannian metric on $\tilde{N}$, at $(q, v) \in \tilde{N} \times T_q\tilde{N}$ is [2,6]

$$\begin{aligned} \mathrm{Hess}\, f_q(v, v) &= \ell\{\langle \nabla X(\ell), v\rangle + \langle \dot{\gamma}(\ell), S_q(v, v)\rangle\} \\ &\geqslant \ell\{k_c(\ell)\cdot |v|^2 - |S_q(v, v)|\} \end{aligned} \quad (1)$$

where $\gamma\colon [0,\ell] \to M$ is the unique normal geodesic joining p to q ($\ell = d_p(q) \in]0,r]$), X the unique Jacobi field along γ in M satisfying the boundary condition $(X(0), X(\ell)) = (0, v)$, ∇X the covariant derivative of X along γ on $(M, \langle\ ,\ \rangle)$ and S the second fundamental form of $\tilde{N}$. As the manifold $\tilde{N}$ is compact, we get a point q in $\tilde{N}$ where f achieves a maximum. So Hess $f_q(v,v) \leqslant 0$ for all $v \in T_q\tilde{N}$ that is

$$\forall v \in T_q\tilde{N}, \quad |S_q(v,v)| \geqslant k_c(\ell)\cdot|v|^2 \geqslant k_c(r)\cdot|v|^2, \tag{2}$$

the last inequality following from the decrease of k_c. Among all unit tangent vectors u to $\tilde{N}$ at q, let u_1 be one which makes $|S_q(u,u)|$ minimal. According to (2), we have $|S_q(u_1,u_1)| \geqslant k_c(r) > 0$ and so the kernel of the linear map $S_q(u_1,\cdot)\colon T_q\tilde{N} \to (T_q\tilde{N})^{\perp}$ is $(n-1)$-dimensional. If $\{u_2,\dots,u_n\}$ is an orthonormal basis of principal vectors of this kernel, the Otsuki Lemma [10] asserts that $\{u_1,\dots,u_n\}$ is an orthonormal basis of principal vectors of $T_q\tilde{N}$. In view of the Gauss formula, the Otsuki Lemma and formula (2), we obtain

$$\begin{aligned}
\mathrm{Ric}_q(u_1) &= \mathrm{Ric}_q^M(u_1) - K_q^M(u_1,\nu) + \sum_{i=2}^{n}\langle S_q(u_1,u_1), S_q(u_i,u_i)\rangle \\
&\geqslant \mathrm{Ric}_q^M(u_1) - K_q^M(u_1,\nu) + \sum_{i=2}^{n}|S_q(u_1,u_1)|^2 \\
&\geqslant \mathrm{Ric}_q^M(u_1) - K_q^M(u_1,\nu) + (n-1)k_c^2(r).
\end{aligned}$$

The hypothesis on the Ricci curvature of N implies that all this inequalities are in fact equalities and so $\ell = r$ and $S_q(u_i,u_i) = k_c(r)\nu$ for all $i \in \{1,\dots,n\}$, that is q is an umbilical point of $\tilde{N}$ lying in $S_p(r)$.

Step 2. We claim that the function f is subharmonic in an open neighborhood of q in $\tilde{N}$.

Proof of this claim: As $n \geqslant 3$, let $\mathcal{U}$ be an open neighborhood of q in $\tilde{N}$ in which the principal curvatures functions $(k_i)_{1\leqslant i\leqslant n}$ of $\tilde{N}$ satisfy

$$\forall i \in \{1,\dots,n\}, \forall q_1 \in \mathcal{U}, 0 < k_i(q_1) < (n-1)k_c(r).$$

Assume that there exists a point q_1 in $\mathcal{U}$ with $\Delta f(q_1) < 0$. By taking the trace of the Hessian in (1), we get

$$\begin{aligned}
0 > \Delta f(q_1) &\geqslant \ell_1\{nk_c(\ell_1) - (k_1(q_1) + \cdots + k_n(q_1))\} \\
&\geqslant \ell_1\{nk_c(r) - (k_1(q_1) + \cdots + k_n(q_1))\}
\end{aligned}$$

so $k_1(q_1) + \cdots + k_n(q_1) > nk_c(r)$, where $\ell_1 = d_p(q_1) \in]0,r]$. One of the numbers $k_i(q_1)$, say $k_1(q_1)$, is strictly greater than $k_c(r)$. Let v_1 be a unit principal vector associated to the principal curvature $k_1(q_1)$. We obtain

$$\begin{aligned} \mathrm{Ric}_{q_1}(v_1) &= \mathrm{Ric}^M_{q_1}(u_1) - K^M_{q_1}(v_1, \nu) + k_1(q_1)\{k_2(q_1) + \cdots + k_n(q_1)\} \\ &> \mathrm{Ric}^M_{q_1}(u_1) - K^M_{q_1}(v_1, \nu) + k_1(q_1)\{nk_c(r) - k_1(q_1)\} \\ &= \mathrm{Ric}^M_{q_1}(u_1) - K^M_{q_1}(v_1, \nu) + (n-1)k_c^2(r) + \\ &\quad +\{k_1(q_1) - k_c(r)\}\{(n-1)k_c(r) - k_1(q_1)\} \\ &> \mathrm{Ric}^M_{q_1}(u_1) - K^M_{q_1}(v_1, \nu) + (n-1)k_c^2(r). \end{aligned}$$

This is a contradiction with the hypothesis.

Step 3. By the maximum principle for subharmonic functions, f is constant on $\mathcal{U}$. So, $\{q \in \tilde{N}/d_p(q) = r\}$ is a nonempty closed open set of the connected manifold $\tilde{N}$ and therefore coincides with $\tilde{N}$, that is $i(N) = \tilde{N} \subset S_p(r)$. As $\tilde{N}$ is an n-dimensional compact manifold, $\tilde{N}$ is also a nonempty closed open set of the connected sphere $S_p(r)$. So $i(N) = \tilde{N} = S_p(r)$. As f is constant in $S_p(r)$, the first part of the proof shows that $S_p(r)$ is an umbilical hypersurface of M with principal curvatures equal to $k_c(r)$. Using a standard argument (as in [4]) on the relationship between the differential of $\exp_p$: $T_p\tilde{N} \to \tilde{N}$ and the behaviour of Jacobi fields along radial geodesics emanating from p, we conclude that the interior of $(\overline{B}_p(r))$ is isometric to the ball of radius r in the space form $\tilde{S}_{n+1}(c)$ of constant curvature c. □

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