M-671, Midterm Examination

I- SEMESTER-1430

Department of Mathematics

College of Science, King Saud University

Time 2 hours. Max. Marks-30

Note: Attempt all questions

Q.1. (a) Let $M$ be an $n$-dimensional smooth manifold and $p \in M$. If $u \in T_p M$, show that there exists a smooth vector field $X \in \mathfrak{X}(M)$ such that $X(p) = u$.

(b) Let $\pi : S^2 \rightarrow \mathbb{RP}^2$ be the projection $\pi(p) = [p], p \in S^2$. Let $(U, \varphi)$ be the chart on $S^2$ defined by $U = \{(x, y, z) \in S^2 : z > 0\}$

$$\varphi(x, y, z) = (x, y)$$

and $X \in \mathfrak{X}(S^2)$ be locally expressed on $U$ by

$$X = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$$

where $x^1, x^2$ are local coordinates on $U$. Find a vector field $Y \in \mathfrak{X}(\mathbb{RP}^2)$ that satisfies $d\pi(X)(p) = Y([p])$ for $p \in U$.

Q.2. (a) Let $\{\varphi_t\}$ be a one-parameter group of transformation of the smooth manifold $M$ that induced a vector field $X$. Show that $X$ is $\varphi_t$-related to itself.

(b) Let $\{\varphi_t\}$ be a one-parameter group of transformation of the smooth manifold $M$ that induced a vector field $X$ and $Y \in \mathfrak{X}(M)$. Show that for $p \in M$

$$[X, Y]_p = \lim_{t \to 0} \frac{d\varphi_{-t}(Y(q)) - Y(q)}{t}$$
where $\varphi_i(p) = q$. Use this to show that two vector fields commute if and only if their one-parameter groups of transformations commute.

**Q.3. (a)** Let $(U, \varphi)$ be a chart on a smooth manifold $M$ and $TM$ be the tangent bundle of $M$. Show that the subset $\pi^{-1}(U)$ is diffeomorphic to $\varphi(U) \times \mathbb{R}^n$, where $\pi: TM \to M$ is the projection map.

**Q.3. (b)** Let $M$ be an $n$-dimensional smooth manifold. Explain what do you mean by the exterior derivative operator $d: \Lambda^k(M) \to \Lambda^{k+1}(M)$ and show that this operator satisfies $d \circ d = 0$

(c) Let $\theta \in \Lambda^2(\mathbb{R}^3)$ be

$$\theta = zdx \wedge dy - ydz \wedge dx + xdy \wedge dz$$

find $d\theta$. 
