M-671, Midterm Examination

I- SEMESTER-1430

Department of Mathematics

College of Science, King Saud University

Time 2 hours. **Max. Marks-30**

Note: *Attempt all questions*

Q.1. **(a)** Let M be an n-dimensional smooth manifold and $p \in M$. If $u \in T_pM$, show that there exists a smooth vector field $X \in \mathfrak{X}(M)$ such that $X(p) = u$.

(b) Let $\pi : S^2 \to RP^2$ be the projection $\pi(p) = [p]$, $p \in S^2$. Let (U, φ) be the chart on S^2 defined by $U = \{(x, y, z) \in S^2 : z > 0\}$

$$\varphi(x, y, z) = (x, y)$$

and $X \in \mathfrak{X}(S^2)$ be locally expressed on U by

$$X = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$$

where x^1, x^2 are local coordinates on U. Find a vector field $Y \in \mathfrak{X}(RP^2)$ that satisfies $d\pi(X)(p) = Y([p])$ for $p \in U$.

Q.2. **(a)** Let $\{\varphi_t\}$ be a one-parameter group of transformation of the smooth manifole M that induced a vector field X. Show that X is φ_t-related to itself.

(b) Let $\{\varphi_t\}$ be a one-parameter group of transformation of the smooth manifole M that induced a vector field X and $Y \in \mathfrak{X}(M)$. Show that for $p \in M$

$$[X, Y]_p = \lim_{t \to 0} \frac{d\varphi_{-t}(Y(q)) - Y(q)}{t}$$

where $\varphi_i(p) = q$. Use this to show that two vector fields commute if and only if their one-parameter groups of transformations commute.

Q.3. **(a)** Let (U, φ) be a chart on a smooth manifold M and TM be the tangent bundle of M. Show that the subset $\pi^{-1}(U)$ is diffeomorphic to $\varphi(U) \times R^n$, where $\pi : TM \to M$ is the projection map.

(b) Let M be an n-dimensional smooth manifold. Explain what do you mean by the exterior derivative operator $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ and show that this operator satisfies $d \circ d = 0$

(c) Let $\theta \in \Lambda^2(R^3)$ be

$$\theta = zdx\Lambda dy - ydz\Lambda dx + xdy\Lambda dz$$

find $d\theta$.