M-671, Final Examination  
Department of Mathematics, College of Science  
King Saud University  
Semester-I, 1431 H.

Time: 3 Hours  
Max. Marks-50

Note: Attempt all questions

Q.1. (i) Let $M$ be an $n$-dimensional smooth manifold and $T^*M$ be its cotangent bundle. Show that the projection map $\pi: T^*M \to M$, $\pi(p, \omega_p) = p$, where $p \in M$ and $\omega_p \in T^*_pM$, is a smooth map. [4]

(ii) If $M$ be an $n$-dimensional smooth manifold and $p \in M$, show that the cotangent space $T^*_pM$ is an $n$-dimensional smooth manifold. [3]

(iii) Let $f, g, h \in C^\infty(R^2 - \{0\})$ be

$$f(x, y) = \frac{1}{2}(x^2 - y^2), \quad g(x, y) = \frac{1}{2}(x^2 + y^2), \quad h(x, y) = xy$$

and show that $\left\{ \frac{\partial g \Delta h}{\partial f} \right\}$ is basis for the space $\Lambda^2(R^2 - \{0\})$. [3]

Q.2. (i) Let $M$ be an $n$-dimensional smooth manifold. Show that the space of smooth $n$-forms $\Lambda^n(M)$ is a 1-dimensional vector space over $\mathbb{R}$. Also show that $\Lambda^{n+1}(M) = \{0\}$. [4]

(ii) Give example of two complete vector fields on a smooth manifold $M$ whose sum is not a complete vector field. Explain in detail your answer. [6]

Q.3. (i) Let $\nabla$ be a linear connection on a smooth manifold $M$. If $X, Y \in \mathfrak{X}(M)$ are such that $X(p) = Y(p)$ for a point $p \in M$, then show that for any $Z \in \mathfrak{X}(M)$, $(\nabla_X Z)(p) = (\nabla_Y Z)(p)$. [3]

(ii) Let $\nabla$ be a linear connection on a smooth manifold $M$ and $\sigma: (a, b) \to M$ be a smooth curve. Show that for each $u \in T_{\sigma(t_0)}M$, $t_0 \in (a, b)$ there exists a unique vector field $Y(t)$ parallel along $\sigma$ with respect to $\nabla$ such that $Y(t_0) = u$. Use this to define the parallel translation map between the tangent spaces $T_{\sigma(t_0)}M$ and $T_{\sigma(t)}M$. [4]

(iii) Consider the Euclidean connection $\nabla$ on $R^2$ and the smooth curve $\alpha: R \to R^2$, $\alpha(t) = (cost, \sin t)$. Find a vector field along $\alpha$ that is parallel along $\alpha$ with respect to the Euclidean connection $\nabla$. [3]
Q.4. (i) Consider the vector field \( \psi = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \in \mathfrak{X}(R^3) \) and the Euclidean inner product \( \langle \cdot, \cdot \rangle \) on \( R^3 \). Show that for a unit sphere \( S^2 \subset R^3 \), the restriction of a vector field \( X \in \mathfrak{X}(R^3) \) to \( S^2 \) is a vector field on \( S^2 \) if and only if \( \langle X, \psi \rangle = 0 \). [2]

(ii) Let \( \nabla \) be the Euclidean connection on \( R^3 \). Then show that \( \nabla : \mathfrak{X}(S^2) \times \mathfrak{X}(S^2) \to \mathfrak{X}(S^2) \) defined by
\[
\nabla_X Y = \nabla_X Y + \langle X, Y \rangle \psi, \quad X, Y \in \mathfrak{X}(S^2)
\]
is a linear connection on \( \mathfrak{X}(S^2) \). Find a geodesic on \( S^2 \) with respect to the linear connection \( \nabla \). [4]

(iii) Find a geodesic passing through \((1,1)\) with respect to a linear connection \( \nabla \) on \( R^2 \) which is given by the Christoffel symbols all of which are zero except
\[
\Gamma^1_{11} = 1, \Gamma^2_{22} = -1
\]

Q.5. (i) Let \( \nabla \) be a linear connection on a smooth manifold \( M \) and \( \omega^i_j, \Omega^i_j \) be the connection forms and curvature form on an open set \( U \subset M \). Derive the structure equation
\[
d\omega_j^i = \sum_{k=1}^n \omega_j^k A \omega^i_k + \Omega^i_j
\]
Also write down this structure equation for the Euclidean connection on \( R^n \). [4]

(ii) Let \( M \) be a smooth manifold with linear connection \( \nabla \) and \( \alpha_{X_p} : (-\epsilon, \epsilon) \to M \) be the geodesic of \( \nabla \) with initial conditions \( \alpha_{X_p}(0) = p, \dot{\alpha}_{X_p}(0) = X_p \). Prove that
\[
\alpha_{X_p}(t) = \alpha_{tX_p}(1)
\]
whenever both sides are defined. [3]

(iii) Let \((M, g)\) be a Riemannian manifold and \( \alpha_{X_p} : (-\epsilon, \epsilon) \to M \) be the geodesic with respect to the Riemannian connection \( \nabla \) with initial conditions \( \alpha_{X_p}(0) = p, \dot{\alpha}_{X_p}(0) = X_p \). Then show that the arc length function
\[
L(t) = \int_0^t \| \dot{\alpha}(s) \| \, ds
\]
satisfies
\[
L(t) = \| X_p \| t
\]

[3]