M-671, Final Examination

Department of Mathematics, College of Science

King Saud University

Semester-I, 1431 H.

Time: 3 Hours **Max. Marks-50**

Note: *Attempt all questions*

Q.1. (i) Let M be an n-dimensional smooth manifold and T^*M be its cotangent bundle. Show that the projection map $\pi : T^*M \to M$, $\pi(p, \omega_p) = p$, where $p \in M$ and $\omega_p \in T_p^*M$, is a smooth map. [4]

(ii) If M be an n-dimensional smooth manifold and $p \in M$, show that the cotangent space T_p^*M is an n-dimensional smooth manifold. [3]

(iii) Let $f, g, h \in C^\infty(R^2 - \{0\})$ be

$$f(x, y) = \frac{1}{2}(x^2 - y^2), \quad g(x, y) = \frac{1}{2}(x^2 + y^2), \quad h(x, y) = xy$$

and show that $\left\{ \frac{dg \Lambda dh}{2f} \right\}$ is basis for the space $\Lambda^2(R^2 - \{0\})$. [3]

Q.2. (i) Let M be an n-dimensional smooth manifold. Show that the space of smooth n-forms $\Lambda^n(M)$ is a 1-dimensional vector space over R. Also show that $\Lambda^{n+1}(M) = \{0\}$. [4]

(ii) Give example of two complete vector fields on a smooth manifold M whose sum is not a complete vector field. Explain in detail your answer. [6]

Q.3. (i) Let ∇ be a linear connection on a smooth manifold.M. If $X, Y \in \mathfrak{X}(M)$ are such that $X(p) = Y(p)$ for a point $p \in M$, then show that for any $Z \in \mathfrak{X}(M)$, $(\nabla_X Z)(p) = (\nabla_Y Z)(p)$. [3]

(ii) Let ∇ be a linear connection on a smooth manifold.M and $\sigma : (a, b) \to M$ be a smooth curve. Show that for each $u \in T_{\sigma(t_0)}M$, $t_0 \in (a, b)$ there exists a unique vector field $Y(t)$ parallel along σ with respect to ∇ such that $Y(t_0) = u$. Use this to define the parallel translation map between the tangent spaces $T_{\sigma(t_0)}M$ and $T_{\sigma(t)}M$. [4]

(iii) Consider the Euclidean connection ∇ on R^2 and the smooth curve $\alpha : R \to R^2$, $\alpha(t) = (\cos t, \sin t)$. Find a vector field along α that is parallel along α with respect to the Euclidean connection ∇. [3]

Q.4. (i) Consider the vector field $\psi = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \in \mathfrak{X}(R^3)$ and the Euclidean inner product $\langle , \rangle$ on R^3. Show that for a unit sphere $S^2 \subset R^3$, the restriction of a vector field $X \in \mathfrak{X}(R^3)$ to S^2 is a vector field on S^2 if and only if $\langle X, \psi \rangle = 0$. [2]

(ii) Let $\overline{\nabla}$ be the Euclidean connection on R^3. Then show that $\nabla : \mathfrak{X}(S^2) \times \mathfrak{X}(S^2) \to \mathfrak{X}(S^2)$ defined by

$$\nabla_X Y = \overline{\nabla}_X Y + \langle X, Y \rangle \psi, \quad X, Y \in \mathfrak{X}(S^2)$$

is a linear connection on $\mathfrak{X}(S^2)$. Find a geodesic on S^2 with respect to the linear connection ∇. [4]

(iii) Find a geodesic passing through $(1, 1)$ with respect to a linear connection ∇ on R^2 which is given by the Christoffel symbols all of which are zero except

$$\Gamma^1_{11} = 1, \, \Gamma^2_{22} = -1$$

[4]

Q.5. (i) Let ∇ be a linear connection on a smooth manifold M and ω^i_j, Ω^i_j be the connection forms and curvature form on an open set $U \subset M$. Derive the structure equation

$$d\omega^i_j = \sum_{k=1}^{n} \omega^k_j \Lambda \omega^i_k + \Omega^i_j$$

Also write down this structure equation for the Euclidean connection on R^n. [4]

(ii) Let M be a smooth manifold with linear connection ∇ and $\alpha_{X_p} : (-\epsilon, \epsilon) \to M$ be the geodesic of ∇ with initial conditions $\alpha_{X_p}(0) = p$, $\dot{\alpha}_{X_p}(0) = X_p$. Prove that

$$\alpha_{X_p}(t) = \alpha_{tX_p}(1)$$

whenever both sides are defined. [3]

(iii) Let (M, g) be a Riemannian manifold and $\alpha_{X_p} : (-\epsilon, \epsilon) \to M$ be the geodesic with respect to the Riemannian connection ∇ with initial conditions $\alpha_{X_p}(0) = p$, $\dot{\alpha}_{X_p}(0) = X_p$. Then show that the arc length function

$$L(t) = \int_0^t \left\| \dot{\alpha}(s) \right\| ds$$

satisfies

$$L(t) = \|X_p\| \, t$$

[3]