Minimal hypersurfaces in $\mathbb{R}^n$ as regular values of a function

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ABSTRACT: In this paper we prove that if $M = f^{-1}(0)$ is a minimal hypersurface of $\mathbf{R}^n$, where $f : V \subset \mathbf{R}^n \to \mathbf{R}$ is a smooth function defined on a open set V, then f must satisfy the equation $|\nabla f|^2 \Delta f = \frac{1}{2}\langle \nabla |\nabla f|^2, \nabla f\rangle$ for every $x \in M$. We will also prove that if M is the zero level set of a homogeneous 2 polynomial, then M must be a Clifford minimal hypersurface.

§1 Introduction and preliminaries:

In this paper we will consider hypersurfaces $M \subset \mathbf{R}^n$ that are level sets of functions, i.e. we will assume that $M = \{x \in V : f(x) = 0\}$ where $f : V \to \mathbf{R}$ is a smooth function defined in an open set of $\mathbf{R}^n$ and $|\nabla f(x)| \neq 0$ for all $x \in M$. For these hypersurfaces, we have that the Gauss map can be written as $\nu(x) = |\nabla f(x)|^{-1}\nabla f(x)$ for all $x \in M$. Clearly, the tangent space of M at a point x is the space of vectors $v \in \mathbf{R}^n$ such that $\langle v, \nabla f(x)\rangle = 0$. Notice that the mean curvature of M at x is given by

$$-\sum_{i=1}^{n-1} \langle d\nu_x(v_i), v_i\rangle \tag{1}$$

where $\{v_1, \dots, v_{n-1}\}$ is an orthonormal bases of the vector space T_xM.

I would like to mention some elementary facts about real value functions on $\mathbf{R}^n$ that will be used later on.

Lemma 1.1: *If $f : V \to \mathbf{R}$ is a smooth function defined in an open set of $\mathbf{R}^n$, then*

(a) $\Delta f(x) = \frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2} = \sum_{i=1}^n \langle Hess(f)_x(w_j), w_j\rangle$ *where $\{w_1, \dots, w_n\}$ is any orthonormal bases of $\mathbf{R}^n$ and $Hess(f)$ is the $n \times n$ hessian matrix of f.*

(b) $\langle Hess(f)\nabla f, \nabla f\rangle = \frac{1}{2}\langle \nabla|\nabla f|^2, \nabla f\rangle$

(c) $\frac{d\nabla f(\alpha(t))}{dt} = Hess(f)_{\alpha(t)}\alpha'(t)$ *for any smooth curve $\alpha : (a, b) \to V$.*

Proof: (a) holds true because $\Delta f(x)$ is the trace of the matrix $\text{Hess}(f)_x$, and the trace of a matrix is invariant under change of bases. (b) is a direct computation and (c) follows from the chain rule. ■

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§2. Main result: In this section we will state and prove one of the main results of this paper.

Theorem 2.1: *Let $M = \{x \in V : f(x) = 0\}$ where $f : V \to \mathbf{R}$ is a smooth function defined in an open set of $\mathbf{R}^n$ with $|\nabla f(x)| \neq 0$ for all $x \in M$. M is minimal if and only if $|\nabla f|^2 \Delta f = \frac{1}{2}\langle \nabla |\nabla f|^2, \nabla f\rangle$ for every $x \in M$*

Proof: We are going to compute the mean curvature H of M in terms of the function f and its partial derivatives. Let us start computing $\langle d\nu_x(v), v\rangle$ for any $v \in T_xM$. Let us take a smooth curve $\alpha : (-\epsilon, \epsilon) \to M$ such that $\alpha(0) = x$ and $\alpha'(0) = v$. We have that

$$\begin{aligned}
\langle d\nu_x(v), v\rangle =&\langle \frac{d\nu(\alpha(t))}{dt}\Big|_{t=0}, v\rangle \\
=&\langle \frac{d|\nabla f(\alpha(t))|^{-1}\nabla f(\alpha(t))}{dt}\Big|_{t=0}, v\rangle \\
=&\frac{d|\nabla f(\alpha(t))|^{-1}}{dt}\Big|_{t=0}\langle \nabla f(x), v\rangle + |\nabla f(x)|^{-1}\langle \frac{d\nabla f(\alpha(t))}{dt}\Big|_{t=0}, v\rangle \\
=&0 + |\nabla f(x)|^{-1}\langle \text{Hess}(f)_x v, v\rangle = |\nabla f(x)|^{-1}\langle \text{Hess}(f)_x v, v\rangle
\end{aligned}$$

Now, if $\{v_1, \dots, v_{n-1}\}$ is an orthonormal bases of T_xM, then by the equation (1) in section 1, we get that,

$$\begin{aligned}
H =& -\sum_{i=1}^{n-1} \langle d\nu_x(v_i), v_i\rangle = -|\nabla f(x)|^{-1}\langle \text{Hess}(f)_x v_i, v_i\rangle \\
=&|\nabla f(x)|^{-1}(-\Delta f(x) + \langle \text{Hess}(f)_x |\nabla f(x)|^{-1}\nabla f(x), |\nabla f(x)|^{-1}\nabla f(x)\rangle) \\
=&|\nabla f(x)|^{-1}(-\Delta f(x) + |\nabla f(x)|^{-2}\frac{1}{2}\langle \nabla |\nabla f|^2, \nabla f\rangle)
\end{aligned}$$

Therefore we get that M is minimal, if and only if, for every $x \in M$, we have that $|\nabla f(x)|^2 \Delta f(x) = \frac{1}{2}\langle \nabla |\nabla f|^2, \nabla f\rangle$. ∎

Example 2.1: (Clifford minimal cones) Let k and l be two positive integers such that $k + l = n - 2$, and let $f : \mathbf{R}^n \setminus \{\mathbf{0}\} \to \mathbf{R}$ be the function given by

$$f(x) = f(x_1, \dots, x_n) = k(x_1^2 + \dots + x_{l+1}^2) - l(x_{1+2}^2 + \dots + x_n^2)$$

Let us check that $M_{lk} = f^{-1}(0)$ is a minimal hypersurface. A direct computation shows that

$$\begin{aligned}
\nabla f(x) =&2(kx_1, \dots, kx_{l+1}, -lx_{l+2}, \dots, -lx_n) \\
|\nabla f(x)|^2 =&4k^2(x_1^2 + \dots + x_{l+1}^2) + 4l^2(x_{l+2}^2 + \dots + x_n^2) \\
\nabla |\nabla f(x)|^2 =&8(k^2x_1, \dots, k^2x_{l+1}, l^2x_{l+2}, \dots, ^2 lx_n) \\
\frac{1}{2}\langle \nabla |\nabla f|^2, \nabla f\rangle =&8k^3(x_1^2 + \dots + x_{l+1}^2) - 8l^3(x_{l+2}^2 + \dots + lx_n^2) \\
\Delta f(x) =&2k(l + 1) - 2l(k + 1) = 2k(k - l)
\end{aligned}$$

Therefore, we have that if $x \in M$, i.e. if $k(x_1^2 + \ldots + x_{l+1}^2) = l(x_{1+2}^2 + \ldots + x_n^2)$, then,

$$
\begin{aligned}
|\nabla f(x)|^2 \Delta f =& 2(k-l)(4k^2 - 4lk)(x_1^2 + \ldots + x_{l+1}^2)\\
=& 8k(k^2 - l^2)(x_1^2 + \ldots + x_{l+1}^2)\\
=& 8k^3 x_1^2 + \ldots + x_{l+1}^2) - 8l^3(x_{1+2}^2 + \ldots + x_n^2)\\
=& \frac{1}{2}\langle \nabla |\nabla f|^2, \nabla f\rangle
\end{aligned}
$$

We will say that M is a *Clifford minimal cone* if $M = M_{kl}$ up to a rigid motion in $\mathbf{R}^n$.

Let us assume now that (N, g) is a riemannian n dimensional manifold, V is an open subset of N and $f : V \to \mathbf{R}$ is a smooth function such that $M = f^{-1}(0)$ is a hypersurface of N, i.e. 0 is a regular value of f. In this case we will denote by $\text{Hess}(f)_x : T_xM \times T_xM \to \mathbf{R}$ the bilinear form defined by $\text{Hess}(f)_x(v, w) = \langle D_v \nabla f, w\rangle$, where D is the Levi Civita connection on N.

The exact same proof of the previous theorem gives us the following result.

Theorem 2.2: *Let N be a riemannian manifold and let $M = \{x \in V : f(x) = 0\}$ where $f : V \to \mathbf{R}$ is a smooth function defined in an open set of N with $|\nabla f(x)| \neq 0$ for all $x \in M$. M is minimal if and only if $|\nabla f|^2 \Delta f = \frac{1}{2}\langle \nabla |\nabla f|^2, \nabla f\rangle$ for every $x \in M$.*

§3 Minimal hypersurfaces of $\mathbf{R}^n$ given by quadratic form

In this section we characterize the minimal Clifford cones as the only hypersurfaces that are level sets of quadratic forms. More precisely we prove,

Theorem 3.1: *Let $f : \mathbf{R}^n \setminus \{\mathbf{0}\} \to \mathbf{R}$ be the function defined by $f(x) = \langle Bx, x\rangle$, where B is a $n \times n$ symmetric matrix and take $M = \{x \in \mathbf{R}^n : \langle Bx, x\rangle = 0\} \setminus \{\mathbf{0}\} = f^{-1}(0)$. the value 0 is a regular value of f and M is a minimal hypersurface if and only if M is a Clifford minimal cone.*

Before we prove this theorem we will need the following lemma

Lemma 3.1: *let B be an invertible symmetric matrix with both, positive and negative eigenvalues. If C is a matrix that commutes with B such that $\langle Cx, x\rangle = 0$ always that $\langle Bx, x\rangle = 0$, then $C = \lambda B$ for some real number λ.*

Proof: Since B and C commutes, after an orthogonal change of coordinates, we can assume that

$$
C = \begin{pmatrix} c_1 & & & & & \\ & \ddots & & & & \\ & & c_r & & & \\ & & & c_{r+1} & & \\ & & & & \ddots & \\ & & & & & c_n \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & & & & & \\ & \ddots & & & & \\ & & b_r & & & \\ & & & b_{r+1} & & \\ & & & & \ddots & \\ & & & & & b_n \end{pmatrix}
$$

where $0 < r < n$, $b_1, \dots, b_r$ are positive real numbers and $b_{r+1}, \dots, b_n$ are negative real numbers. Let us denote $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ the canonical bases for $\mathbf{R}^n$ and for $1 \le i \le r$ and $r < j \le n$ we will denote $x_{ij} = \sqrt{-b_j} e_i + \sqrt{b_i} e_j$. Notice that $\langle Bx_{ij}, x_{ij} \rangle = 0$, therefore $\langle Cx_{ij}, x_{ij} \rangle = 0$, i.e.

$$
-b_j c_i + b_i c_j = 0
$$

from the equation above we get that $c_i = \frac{c_n}{b_n} b_i$ for $1 \le i \le r$ and for $r < j \le n$ we have, using the expression for c_1, that

$$
c_j = \frac{c_1}{b_1} b_j = \frac{c_n}{b_n} b_1 \frac{1}{b_1} b_j = \frac{c_n}{b_n} b_j
$$

Therefore $C = \frac{c_n}{b_n} B$, this completes the proof. ∎

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: A Direct computation shows that $\nabla f(x) = 2Bx$. Since we are assuming that 0 is a regular value, then $\nabla f(x) \neq \mathbf{0}$ for all $x \in M$, in particular $Bx_0 \neq \mathbf{0}$ for every $x_0 \neq \mathbf{0}$, because if $Bx_0 = \mathbf{0}$ for some $x_0 \neq \mathbf{0}$ we would have that $x_0 \in M$ and $\nabla f(x_0) = \mathbf{0}$. Therefore B is an invertible matrix. We also have that, since $M \neq$, then B must have positive eigenvalues and negative eigenvalues. We have that

$$
|\nabla f(x)|^2 = \langle 2Bx, 2Bx \rangle = 4 \langle Bx, Bx \rangle = 4 \langle B^2 x, x \rangle
$$

Since B^2 is symmetric, then $\nabla |\nabla f|^2(x) = 8B^2 x$. A direct computation shows that $\Delta f = 2\text{trace}(B)$. Using Theorem 2.1, we have that M is minimal if and only if $|\nabla f|^2 \Delta f = \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle$ for every $x \in M$, i.e. if, for every $x \neq \mathbf{0}$ such that $\langle Bx, x \rangle = 0$ we have that

$$
4 \langle B^2 x, x \rangle (2\text{trace}(B) = \frac{1}{2} \langle 8B^2 x, 2Bx \rangle = 8 \langle B^3 x, x \rangle
$$

In others words, if we define $C = \text{trace}(B) B^2 - B^3$, we have that M is minimal if and only if $\langle Cx, x \rangle = 0$ for every x such that $\langle Bx, x \rangle = 0$. Using Lemma 3.1 we concluded that there exists a real number a such that $\text{trace}(B) B^2 - B^3 = aB$. Since B is an invertible matrix, we get that B satisfies the following polynomial equation

$$B^2 - \text{trace}(B)B + aI = 0 \tag{2}$$

Therefore B can only have two eigenvalues. Since B has negative and positive eigenvalues, we can assume that the eigenvalues of B are $\lambda_1 > 0$ with multiplicity $r \geq 1$ and $\lambda_2 < 0$ with multiplicity $n - r \geq 1$. Notice that $\text{trace}(B) = r\lambda_1 + (n-r)\lambda_2$. The equation (2) is equivalent to the following system of equations for λ_1, λ_2 and a.

$$\begin{aligned}\lambda_1^2 - (r\lambda_1 + (n-r)\lambda_2)\lambda_1 + a =&(1-r)\lambda_1^2 - (n-r)\lambda_1\lambda_2 + a = 0\\ \lambda_2^2 - (r\lambda_1 + (n-r)\lambda_2)\lambda_2 + a =&-(n-r-1)\lambda_2^2 - r\lambda_1\lambda_2 + a = 0\end{aligned}$$

combining these two equations we get

$$(1-r)\lambda_1^2 - (n-2r)\lambda_1\lambda_2 + (n-r-1)\lambda_2^2 \tag{3}$$

From this equation we get that $r = 1$ or $r = n-1$ implies that $\lambda_1 = \lambda_2$ which is impossible because $\lambda_1\lambda_2 < 0$. Therefore $1 < r < n-1$. From equation (3) we get that $t = \frac{\lambda_2}{\lambda_1}$ satisfies the equation

$$(1-r) - (n-2r)t + (n-r-1)t^2$$

Therefore $t = 1$ or $t = \frac{r-1}{n-r-1}$, since we have that t must be negative, then t cannot be 1. Therefore up to a constant we may take $\lambda_1 = n-r-1$ and $\lambda_2 = r-1$. i.e. Up to a rigid motion $f(x) = \langle Bx, x\rangle$ must be a multiple of the function given in the example 2.1. This implies that M must be a Clifford minimal cone. ■

Remark on the construction of minimal hypersurfaces using homogeneous polynomials of degree k: Let $f : \mathbf{R}^n \setminus \{\mathbf{0}\} \to \mathbf{R}$ be a homogeneous polynomial of degree k such that $f^{-1}(0) = M$ is not empty and such that for every $x \in M$, $\nabla f(x) \neq \mathbf{0}$. By theorem 2.1 we have that M is minimal if and only if

$$g(x) = |\nabla f|^2 \Delta f - \frac{1}{2}\langle \nabla |\nabla f|^2, \nabla f\rangle = 0 \tag{4}$$

for every x such that $f(x) = 0$. Notice that the left hand side of the equation (4) is a homogeneous polynomial of degree $3k-4$, also notice that if $g(x) = h(x)f(x)$ for some homogeneous polynomial h of degree $2k-4$, then M will be minimal.

It is easy to prove that the veracity of the conjecture:

Conjecture: *Let $f : \mathbf{R}^n \setminus \{\mathbf{0}\} \to \mathbf{R}$ be a homogeneous polynomial of degree k such that $f^{-1}(0) = M$ is not empty and such that for every $x \in M$, $\nabla f(x) \neq \mathbf{0}$. If $g(x)$ is a polynomial of degree m with $m \geq k$ such that $g(x) = 0$ for every $x \in M$, then there exists a homogeneous polynomial h of degree $m-k$ such that $g(x) = h(x)f(x)$.*

implies the following result:

"Let $M = f^{-1}(0) \neq \emptyset$ *where* $f : \mathbf{R}^n \setminus \{\mathbf{0}\} \to \mathbf{R}$ *be a homogeneous polynomial.* M *is minimal if and only if*

$$|\nabla f|^2 \Delta f - \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle = hf$$

for some homogeneous polynomial h*".*

So far the only known examples of these minimal hypersurfaces are the isoparametric minimal hypersurfaces, the degree of f in this examples are $k = 1, 2, 3, 4, 6$. Notice that Lemma 2.1 proves the conjecture when $k = 2$.

Bibliography

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