QUOTIENT SPACES

Given two topological spaces $X$ and $Y$ and a surjective function $f : X \to Y$, we are familiar with the terms ‘continuous’, as well as ‘open’, ‘closed’ maps. We shall introduce a concept which is stronger than ‘continuity’ and weaker than ‘homeomorphism’, such onto functions will be called as ‘Quotient maps’. Formally we define

**Definition:** Let $X$ and $Y$ be two topological spaces and $p : X \to Y$ be a surjective map. The map $p$ is said to be a quotient map provided a subset $U$ of $Y$ is open in $Y$ if and only if $p^{-1}(U)$ is open in $X$.

Since the map $p : X \to Y$ is surjective for any $B \subseteq Y$, we have

$$p^{-1}(Y - B) = X - p^{-1}(B)$$

and consequently we get the equivalent form of the definition of a quotient map that requires ‘subset $U$ of $Y$ is closed in $Y$ if and only if $p^{-1}(U)$ is closed in $X$’.

**Remark:** It is clear from the definition that a quotient map is certainly continuous, and that the projections $p_i : X_1 \times X_2 \to X_i$ are quotient maps, where $X_1 \times X_2$ is the product space of the topological spaces $X_1$ and $X_2$.

**Remark:** We shall consider a topological space $X$ and a set $Y$. Suppose we are given a surjective map $p : X \to Y$. Then we can define a collection $\tau_Y = \{ U \subseteq Y : p^{-1}(U) \text{ is open in } X \}$ which can be easily shown to define a topology on $Y$ with respect to which $p : X \to Y$ becomes a quotient map. It is a straightforward to check the following:

**Exercise1.** If $\tau$ is any topology on $Y$ such that $p : X \to Y$ is continuous with respect to $\tau$, then $\tau \subseteq \tau_Y$.

**Exercise2.** The topology $\tau_Y$ is unique, that is if $\tau$ is a topology on $Y$ such that $p : X \to Y$ is a quotient map with respect to $\tau$, then $\tau = \tau_Y$.

**Exercise3.** If $f : X \to Y$ is an open(or closed) continuous surjective map, then $f$ is a quotient map. However the converse is not true.

Proposition3 below gives the condition under which the converse is true.
**Some properties of a quotient map:** We summarize some important properties shared by a quotient map in the following propositions:

**Proposition 1:** Let \( f : X \to Y \) be a continuous map. If there exists a continuous function \( g : Y \to X \) such that \( f \circ g = i_Y \) is the identity map of \( Y \), then \( f \) is a quotient map.

**Proposition 2:** Composition of two quotient maps is a quotient map.

**Proposition 3:** Let \( p : X \to Y \) be a quotient map. Then \( p \) is an open (respectively closed) map if and only if for each open (respectively closed) set \( U \subseteq X \), the set \( p^{-1}(p(U)) \) is open (respectively closed) set in \( X \).

**Proposition 4:** Let \( p : X \to Y \) be a quotient map and \( f : Y \to Z \) be any function. Then the function \( f \) is continuous if and only if \( f \circ p \) is continuous.

**Remark:** Given a topological space \( X \) and a set \( Y \), we saw in Exercise 2 above that a surjective map \( p : X \to Y \) defines a unique topology \( \tau_Y \) on \( Y \) with respect to which \( p \) is a quotient map. This topology \( \tau_Y \) on \( Y \) defined by a surjective map \( p \) and is called the *Quotient topology* on \( Y \) and \( Y \) with this topology is called the *Quotient space*. In light of this definition we see that Proposition 4 above gives a characterization for a continuous function on a quotient space.

**Exercise set-1**

1. Give examples for each of the following:
   (a) A continuous map which is not a quotient map.
   (b) A quotient map which is not open
   (c) A quotient map which is not closed
   (d) A quotient map which is neither closed nor open
   (e) An open map which is not a homeomorphism
   (f) A closed map which is not a homeomorphism

2. Show that a continuous map from a compact space onto a Hausdorff space is a quotient map. Show that the condition ‘Hausdorff’ can not be omitted.
3. Let \( X = [0, 1] \) be subspace of \( R \) and \( S^1 = \{(x, y) \in R^2 : x^2 + y^2 = 1\} \) be the unit sphere as subspace of \( R^2 \). Define \( f : [0, 1] \rightarrow S^1 \) by \( f(t) = e^{2\pi it} \). Show that \( f \) is a quotient map.

4. Let \( (X, d) \) and \( (Y, d') \) be two metric spaces and \( f : X \rightarrow Y \) be a distance preserving function, that is, \( d'(f(x), f(y)) = d(x, y) \) holds. Show that \( f \) is a quotient map.

5. Is a one to one quotient map a homeomorphism?

6. Show that the restriction of a quotient map to an open subset need not be a quotient map.

7. Let \( \pi_1 : R^2 \rightarrow R \) be the projection on the first coordinate and \( Y \) be the subspace \( (\{0\} \times R) \cup (R \times \{0\}) \) of \( R^2 \). Let \( p \) be the restriction of \( \pi_1 \) to \( Y \). Show that \( p \) is a quotient map. Is \( p \) an open map? is it a closed map?

8. Let \( \pi_1 : R^2 \rightarrow R \) be the projection on the first coordinate and \( \beta = \{\pi_1^{-1}\{(a, b) : a, b \in R\}\} \) be the collection of subsets of \( R^2 \). Show that \( \beta \) is a basis for a topology \( \tau \) on \( R^2 \). Compare \( \tau \) with the standard topology on and denote \( X = (R^2, \tau) \). Is \( \pi_1 : X \rightarrow R \) a quotient map? Is \( X \) Hausdorff?

9. Show that the map \( p : R^2 \rightarrow R \) defined by \( p(x, y) = x^2 + y^2 \) is a quotient map.

10. Let \( Y \) be the subspace of \( R^2 \) as given in exercise 7. Define \( p : R^2 \rightarrow Y \) by

\[
p(x, y) = \begin{cases} (x, 0), & x \neq 0 \\ (0, y), & x = 0 \end{cases}
\]

Is \( p \) a quotient map? Is \( p \) continuous?

11. Is the quotient space of a locally compact space necessarily locally compact? What happens if ‘locally compact’ is replaced by ‘compact’ in this question?

**Equivalence relation and quotient map:** We will use the properties of a surjection \( f : X \rightarrow Y \) to define an equivalence relation \( \sim \) on \( X \) as follows:

For \( x, y \in X \), \( x \sim y \) if and only if \( f(x) = f(y) \).

It can be easily shown that \( \sim \) is an equivalence relation on \( X \) and we shall denote by \( X^* = \{[x] : x \in X\} \) where \( [x] = \{y \in X : y \sim x\} \) is the equivalence class of \( x \). Moreover we have \( Y = X^* \) in the sense that if \( y = f(x) \) then \( y = [x] \), and \( f : X \rightarrow X^* \) satisfies \( f(x) = [x] \) that is it becomes the natural projection corresponding to the equivalence relation induced by it on \( X \). Conversely given an equivalence
relation ~ on a nonempty set X we have the set X* of equivalence classes of X and the natural projection p : X → X*. Thus an equivalence relation ~ on a topological space X gives the quotient space X* (the set of equivalence classes of X) and the natural projection p : X → X* becomes the quotient map. Note that X* is a partition of X and that any partition X* of X also gives an equivalence relation on X.

Remark: Suppose that we have a partition X* of a topological space X. Then we can describe the quotient topology on X*, which is a collection of the subsets U ⊂ X* containing those equivalence classes such that the set p⁻¹(U) is just the union of the equivalence classes belonging to U. Thus a typical open set in X* is a collection of equivalence classes whose union is an open set in X.

Theorem: Let X, Y be topological spaces and f : X → Y be continuous surjection. Let X* be the collection \{f⁻¹(y) : y ∈ Y\} of subsets (fibers) of X. The following hold for the quotient space X* :

(a) If Y is Hausdorff, then so is X*
(b) The map h : X* → Y defined by h([x]) = f(x), x ∈ X is a homeomorphism if and only if f is a quotient map.

Examples: 1. Consider the rectangle X = [0, 1] × [0, 1] as a subspace of R². Let X* be the partition of X consisting of the one point sets \{(x, y)\}, 0 < x < 1, 0 < y < 1, the two point sets \{(x, 0), (x, 1)\}, 0 < x < 1, \{(0, y), (1, y)\}, 0 < y < 1 and the four point set \{(0, 0), (0, 1), (1, 0), (1, 1)\}. If we sketch the figure of the quotient space X* we see that it is a torus, and describe the quotient topology by showing the typical sets p⁻¹(U) for an open set U ⊂ X*.

2. Let X = \{(x, y) ∈ R² : x² + y² ≤ 1\} be the closed unit ball with subspace topology. Let X* be a partition of X consisting of all one point sets \{(x, y)\}, x² + y² < 1 and the set S¹ = \{(x, y) : x² + y² = 1\}. The quotient space X* is homeomorphic to the unit 2-sphere S² ⊂ R³. Describe a typical open set U on X* and corresponding p⁻¹(U).

3. Consider X = [0, 1] × [0, 1] as a subspace of R² and the partition X* of X consisting of all singletons \{(x, y)\}, 0 < x < 1, 0 < y < 1 , and the two point sets \{(0, y), (1, 1 − y)\}. The quotient space X* is the Mobius band. Draw the figure and analyze the space.
4. Consider $X = [0, 1] \times [0, 1]$ as a subspace of $\mathbb{R}^2$ and define an equivalence relation $\sim$ on $X$ by
\begin{enumerate}[(i)]
  \item $(x, y) \sim (x, y)$, $x \in (0, 1), y \in (0, 1)$
  \item $(x, 0) \sim (x, 1), x \in [0, 1]$
  \item $(0, y) \sim (1, 1-y), y \in [0, 1]$.
\end{enumerate}
The resulting quotient space $X^*$ is called the Klein bottle. Draw the figure, note that first two steps give a cylinder and then opposite circumferences of this cylinder are joined after giving a twist to this cylinder.

5. Consider the subspace $X = \mathbb{R}^n - \{0\}$ of $\mathbb{R}^n$ and define an equivalence relation $\sim$ on $X$ by $x \sim y$ if and only if $x = ty$, $x, y \in X$, where $t$ is a nonzero real number. The resulting quotient space is denoted by $RP^n$ and is called the $n$-dimensional real projective space. If we replace $\mathbb{R}^n$ by $\mathbb{C}^n$, where $\mathbb{C}$ is the set of complex numbers, the resulting quotient space is denoted by $CP^n$ and is called the $n$-dimensional complex projective space.

\textbf{Exercise set-2}

1. Consider $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ as subspace of $\mathbb{R}^2$. Use above theorem to show that $f : \mathbb{R} \rightarrow S^1$ defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$ is a quotient map.

2. Define a relation $\sim$ on $\mathbb{R}^2$ by requiring that any two points will be related under $\sim$ if they have the same first coordinate. Show that $\sim$ is an equivalence relation and the resulting quotient space is homeomorphic to $\mathbb{R}$.

3. Define a relation $\sim$ on $X = \mathbb{R}$ by requiring that $x \sim y$ if $x - y$ is an integer. Show that the quotient space $X^*$ is homeomorphic to $S^1$.

4. Define a relation $\sim$ on $X = \mathbb{R}^2$ by requiring that $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 + y_2^2 = x_2 + y_1^2$. Show that $\sim$ is an equivalence relation and describe the quotient space $X^*$ by showing that it is homeomorphic to a well known space. What is it?

5. Let $A = (0, 1)$ be the open interval in $X = \mathbb{R}$ and $X^*$ be the partition of $X$ consisting of all singletons $\{x\}, x \in R - (0,1)$ and the set $A$. Describe the quotient topology on $X^*$. Show that any open set in $X^*$ containing either 0 or 1 must contain the set $A$. Is the set $\{A\}$ open in $X^*$? Is $X^*$ Hausdorff?

6. Let $X = [0, 1]$ be the subspace of $\mathbb{R}$ and define $\sim$ on $X$ by $x \sim y$ if $x - y$ is a rational number. Show that the quotient space $X^*$ is not Hausdorff.
7. Let $p : X \to Y$ be a quotient map. Let $Z$ be a topological space and $h : X \to Z$ be a continuous map that is constant on each fiber $p^{-1}(\{y\})$, $y \in Y$. Then show that $h$ induces a continuous map $f : Y \to Z$ such that $f \circ p = h$.

8. Let $X^*$ be the partition of $X = R^2$ consisting of all concentric circles. Show that the quotient space $X^*$ is homeomorphic to the subspace $[0, \infty)$ of $R$.

9. Let $X = \bigcup_{n=0}^{\infty} (R \times \{n\})$ be the subspace of $R^2$ where $n$ is a positive integer and $L'_n$ be the straight line in $R^2$ passing through origin and having slope $n$. Let $Y = \bigcup_{n=0}^{\infty} L'_n$ be a subspace of $R^2$. Consider the map $f : X \to Y$ defined by $f(x,n) = (x,nx)$. Show that $f$ maps a line $L'_n = R \times \{n\}$ on to the line $L'_n$ and is a continuous surjection. Show that $f$ is not a quotient map by showing the quotient topology on $Y$ is not equal to its subspace topology. (Compare with exercise-6 in Exercise set-1)

10. Let $\sim$ be an equivalence relation on the topological space $X$. If the quotient space $X^*$ and each equivalence class $[x], x \in X$ are connected then show that $X$ is connected.

11. Show that a quotient space $X^*$ of a locally connected space $X$ is locally connected.

**Hausdorff criterion:** In exercise-6 above, we have seen that though $X$ is Hausdorff the quotient space $X^*$ is not Hausdorff. This leads to the important question of finding the criterion for the quotient space of a Hausdorff space to be Hausdorff. For this first we introduce some concepts.

**Definition:** An equivalence relation $\sim$ on a topological space $X$ is said to be open if whenever a subset $A \subset X$ is open then the set $[A] = \bigcup_{a \in A} [a]$ is open in the space $X$. Similarly we define a closed relation.

**Lemma:** An equivalence relation $\sim$ on a topological space $X$ is open(closed) if and only if $p : X \to X^*$ is an open(closed) mapping.

**Proof:** Suppose that $p$ is an open mapping and $A \subset X$ be an open subset. Then as $[A] = p^{-1}(p(A))$, and $p(A)$ being an open subset of $X^*$, we see that $[A]$ is an open subset of $X$. Thus the relation $\sim$ is open. Conversely, suppose that the relation $\sim$ is an open relation and $A$ be an open subset of $X$. Then as $[A]$ is open and $[A] = p^{-1}(p(A))$, and $p$ is a quotient map we get that $p(A)$ is open proving that $p$ is an open mapping. The proof for closed relation is similar.
Theorem: Let \( \sim \) be an open equivalence relation on a topological space \( X \). Then \( R = \{(x, y) : x \sim y\} \) is a closed subspace of \( X \times X \) if and only if the quotient space \( X^* \) is Hausdorff.

Proof: Suppose the quotient space \( X^* \) is Hausdorff and that \( x, y \in X \) are such that \( x \) is not related to \( y \) that is \( (x, y) \notin R \). Then as \( p(x) \neq p(y) \), where \( p : X \to X^* \) is the projection, there exist disjoint open sets \( U, V \subset X^* \) containing \( p(x) \) and \( p(y) \) respectively. Let \( U^* = p^{-1}(U) \), and \( V^* = p^{-1}(V) \). Clearly \( x \in U^*, y \in V^* \). If the open set \( U^* \times V^* \) containing the point \( (x, y) \) intersects \( R \), then it will contain a point \( (a, b) \) of \( R \) which satisfies \( a \sim b \Rightarrow p(a) = p(b) \), which will imply that \( p(a) = p(b) \in U \cap V \) contrary to our assumption. Hence \( U^* \times V^* \) does not intersect \( R \). Thus \( X \times X - R \) is open and consequently the set \( R \) is closed in \( X \times X \).

Conversely suppose that \( R \) is closed in \( X \times X \). Then given any distinct pair of points \( p(x), p(y) \in X^* \), there is an open set of the form \( U^* \times V^* \) containing the point \( (x, y) \) and having no point from \( R \) (as \( p(x) \neq p(y) \), that is \( (x, y) \notin R \) and \( X \times X - R \) is open). Put \( U = p(U^*), V = p(V^*) \), which by hypothesis together with above lemma imply that they are open subsets of \( X^* \) containing \( p(x) \) and \( p(y) \) respectively. Suppose \( [a] \in U \cap V \), which will imply the existence of \( b \in U^*, c \in V^* \) such that \( p(b) = p(c) = [a] \Rightarrow (b, c) \in R \). This contradicts the choice of \( U^* \times V^* \). Hence \( U \cap V \) is empty and this proves that \( X^* \) is Hausdorff.

Remark: Note that the quotient space \( X^* \) could be Hausdorff space without \( X \) being Hausdorff. For example \( X = (R^2, \tau) \) in exercise-8 in Exercise set-1 is not Hausdorff yet the quotient space \( X^* \) given by the quotient map \( \pi_1 : X \to \hat{R} \) is homeomorphic to \( R \) and is therefore Hausdorff. In above Theorem we don’t require \( X \) to be Hausdorff. The condition on the relation in above theorem can not be replace by the closed relation see exercise-1 below.

Exercise set-3

1. Consider the projection \( \pi_1 : R^2 \to R \) on the first copy, and the closed set \( F = \{(x, y) \in R^2 : xy = 1\} \) in \( R^2 \). Show that the union \( \bigcup_{x \in R} \pi_1^{-1}(x) \), such that \( \pi_1^{-1}(x) \cap F \) is nonempty, is not closed in \( R^2 \). (Note that \( \pi_1^{-1}(x) \) are equivalence classes) This shows that the relation induced by the function \( \pi_1 \) on \( R^2 \) is not closed. (Compare with exercise-8 in Exercise set-1, and above theorem)
2. Show that the $n$-dimensional real projective space $RP^n$ and the $n$-dimensional complex projective space $CP^n$ are Hausdorff spaces.