CONNECTED SPACES

Among the properties of a topological spaces, connectedness and compactness are important properties. Here we shall study the connectedness and its influence on a topological space.

**Definition:** Let $X$ be a topological space. A separation of $X$ is a pair $U, V$ of disjoint nonempty open subsets of $X$ whose union is $X$. The space $X$ is said to be connected if there does not exist a separation of $X$.

**Remark:** A quick look at the definition suggests that a space $X$ is connected if and only if the only sets which are both open and closed in $X$ are empty set and the space $X$ itself.

**Examples:**
1) The real line $\mathbb{R}$ is connected for if $\mathbb{R} = U \cup V$ is a separation of $\mathbb{R}$. Take $a \in U$, $b \in V$ and assume that $a < b$. Put $W = U \cap [a, b]$, then as $W$ is intersection of two closed sets is a closed subset of $\mathbb{R}$. Let $c = \sup W$. Then as $W$ is closed and bounded $c \in W \subseteq U$. Note that $c \neq b$, as $b \notin W \subseteq U$. Consequently we have $c < b$. Also $(c, b] \cap U$ is empty and consequently $(c, b] \subseteq V$. Taking closure on both sides we see $c \in V = V$. Thus $c$ belongs to both $\bar{U}$ and $V$ a contradiction and this proves that $\mathbb{R}$ is connected.

2) All discrete spaces with more than one point are disconnected, whereas as indiscrete spaces are connected.

3) The subspace $X = \{x \in \mathbb{R} : x \neq 0\}$ is disconnected as well as the subspace $X = \mathbb{R}^2 - \{(x, y) \in \mathbb{R}^2 : y = 0\}$ of $\mathbb{R}^2$ is disconnected.

4) $R_f$ ($\mathbb{R}$ with cofinite topology) is connected (prove!)

For the subspace we have the following:

**Lemma:** If $Y$ is a subspace of $X$, a separation of $Y$ is a pair of disjoint nonempty sets $A$ and $B$ whose union is $Y$, neither of which contains a limit point of the other. The space $Y$ is connected if and only if there exists no separation of $Y$.

**Proof:** (See Munkres p-148)
As an application of above lemma we see that the subspace \( X = \{(x, y) \in \mathbb{R}^2 : y = 0\} \cup \{(x, y) \in \mathbb{R}^2 : x > 0, xy = 1\} \) of \( \mathbb{R}^2 \) is not connected as the two subsets above which describe \( X \) form a separation of \( X \) as described in above lemma.

**Proposition 1.** If the sets \( C \) and \( D \) form a separation of \( X \) and if \( Y \) is connected subset of \( X \), then \( Y \) lies entirely within either \( C \) or \( D \).

**Proposition 2.** The union of a collection of connected sets that have a point in common is connected.

**Proposition 3.** Let \( A \) be a connected subset of \( X \). If \( A \subset B \subset A \), then \( B \) is also connected.

**Proposition 4.** The image of a connected space under a continuous map is connected.

**Proposition 5.** The product of connected spaces is connected.

**Proofs:** (See Munkres pp149-150)

**Exercise set-1**

1) Show that a subset of \( \mathbb{R} \) is connected if and only if it is an interval.

2) Show that the set of rationals \( \mathbb{Q} \) is disconnected in the subspace topology.

3) Is \( \mathbb{R} \) in lower limit topology connected?

4) Let \( X = \mathbb{R} \cup \{\infty\} \) and \( \tau \) be the collection containing empty set and all those subsets of \( X \) which contain \( \infty \). Show that \( \tau \) is a topology on \( X \). What is the subspace topology on \( R \subset X \)? Show that \( X \) is connected while \( R \) is not in this topology. Is connected ness hereditary?

5. Use proposition-2 to show that \( \mathbb{R} \) is connected, and that \( \mathbb{R}^n \) is connected.

6. Show that the \( n \)-sphere \( S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\} \) is connected.

7. Show that a topological space \( X \) is connected if and only if every continuous function \( f : X \to \{0, 1\} \) is constant.
8. Show that a connected space remains connected if its topology is weakened.
9. Show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.
10. If every pair of points of a topological space $X$ can be joined in $X$ by a connected set, then show that $X$ is connected.
11. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $c$ be a real number between $f(a)$ and $f(b)$. Show that there exists a $t \in [a, b]$ such that $f(t) = c$.
12. Show that for a continuous function $f : [0, 1] \to [0, 1]$ there exists an $x \in [0, 1]$ such that $f(x) = x$.
13. Suppose $A$ and $B$ are open subsets of a topological space $X$ for which $A \cup B$ and $A \cap B$ are connected. Show that $A$ and $B$ are connected.

Path Connectedness: There is a slightly stronger notion than connectedness and it is path-connectedness.

Definition: Let $X$ be a topological space. For a pair of points $x, y \in X$, a path in $X$ from $x$ to $y$ is a continuous map $f : [a, b] \to X$ of some closed interval $[a, b] \subset \mathbb{R}$ satisfying $f(a) = x$, $f(b) = y$. A topological space $X$ is said to be path-connected if every pair of points in $X$ can be joined by a path.

It immediately follows from above definition that a path-connected space $X$ is essentially connected. However the converse is not true, as shown in examples 2) and 3).

Examples: 1) $\mathbb{R}$ is path-connected as for any pair $a, b \in \mathbb{R}$, the map $f : [0, 1] \to \mathbb{R}$, defined by $f(t) = (1 - t)a + tb$, is a path joining $a$ and $b$. The same technique yields that $\mathbb{R}^n$ is also path-connected.

2) Comb space: Let $K = \left\{ \frac{1}{n} : n \in \mathbb{Z}_+ \right\}$. Consider the subset of $\mathbb{R}^2$ given by

$C = \{(x, 0) : x \in [0, 1]\} \cup \{(k, y) : k \in K, y \in [0, 1]\} \cup \{(0, y) : y \in [0, 1]\}$

This subspace $C$ of $\mathbb{R}^2$ is called the comb space. The subspace

$D = C - \{(0, y) : y \in (0, 1)\}$

is called the deleted comb space. Note that $C$ is connected and that the set

$\{(x, 0) : x \in [0, 1]\} \cup \{(k, y) : k \in K, y \in [0, 1]\}$
being union of connected sets with common point is connected and has limit point \((0, 1)\). Thus

\[ D = \{(x, 0) : x \in [0, 1]\} \cup \{(k, y) : k \in K, y \in [0, 1]\} \cup \{(0, 1)\} \]

is connected. However the deleted comb space is not path-connected.

We shall show that the point \(p = (0, 1) \in D\) can not be joined to any other point of \(D\) by a path. To prove this consider a path \(f : [a, b] \to D\) in \(D\) with \(f(a) = p\). Since \(\{p\}\) is closed in the Hausdorff space \(D\), \(f^{-1}\{p\}\) is closed in \([a, b]\). If we show that \(f^{-1}\{p\}\) is also open in \([a, b]\), as \([a, b]\) is connected it would mean that \(f^{-1}\{p\} = [a, b]\), which will amount to say that any path in \(D\) starting at \(p\) is a constant path that is it does not join any other point of \(D\). This will prove that there is no path in \(D\) joining any point in \(D - \{p\}\) to \(p\). Thus we have to only show that \(f^{-1}\{p\}\) is open in \([a, b]\).

Take a point \(c \in f^{-1}\{p\}\). Since \(f\) is continuous at \(c\), with \(f(c) = p\), we choose an open ball \(V\) around \(p\) which does not intersect \(-x\)-axis and an open interval \(U = (c - \varepsilon, c + \varepsilon)\) such that \(f(U) \subset V^* = V \cap D\). This is the requirement of the continuity of \(f\) at \(c\). We shall show that \(U \subset f^{-1}\{p\}\), and this will show that \(c\) is an interior point of \(f^{-1}\{p\}\) and consequently that \(f^{-1}\{p\}\) is open. For this choose a point \(q = (\frac{1}{n}, y_0) \in V^*\) such that \(p \neq q\), and a real number \(r\) such that \(\frac{1}{n+1} < r < \frac{1}{n}\). Since continuous image of connected set is connected, \(f(U)\) is connected subset of \(D\), and it does not touch \(-x\)-axis, it does not intersect the vertical line \(x = r\). Now consider the disjoint subsets \((-(\infty, r) \times R)\) and \((r, \infty) \times R\) of \(R^2\). Since \(f(U)\) is connected and contains a point of \((-(\infty, r) \times R)\), it can not contain the point \(q \in (r, \infty) \times R\). This proves that \(f(U) = \{p\}\), that is \(U \subset f^{-1}\{p\}\), which shows that \(c\) is an interior point of \(f^{-1}\{p\}\). Thus the deleted comb space \(D\) is not path-connected.

3) **Topologist’s sine curve:** Consider the subspace

\[ S = \left\{ (x, \sin \frac{1}{x}) : 0 < x \leq 1 \right\} \]

of \(R^2\). Then as the interval \((0, 1]\) is connected and \(f : (0, 1] \to R^2\) defined by \(f(x) = (x, \sin \frac{1}{x})\) is continuous, \(S\) is connected. Therefore the closure \(\overline{S}\) of \(S\) in \(R^2\) is also connected. The subspace \(\overline{S}\) is called the topologist’s sine curve. Note that \((0, 0) \in \overline{S}\) and it can be shown like previous example that no path exists joining \((0, 0)\) to a point \((a, \sin \frac{1}{a}), a > 0\) in \(\overline{S}\). Thus \(\overline{S}\) is connected but not path-connected.
Exercise set-2

1. Show that a nonempty open connected subset of $\mathbb{R}^n$ is path-connected.

2. Let $A, B$ be path-connected subsets of a topological space $X$. If $A \cap B$ is nonempty show that $A \cup B$ is path-connected.

3. Show that $\mathbb{R}^n, n > 1$ is not homeomorphic to $\mathbb{R}$.

4. Is the continuous image of a path-connected space path-connected? What are other such questions?

5. Show that the $n$-sphere $S^n, n > 1$, is path-connected.

Components: In a topological space $X$ we can define a relation by requiring $x \sim y$ if and only if there is a connected subset of $X$ containing both $x$ and $y$. This is an equivalence relation and the equivalence classes of this relation are called the components of $X$. We have the following:

Theorem: The components of a topological space $X$ are connected disjoint subsets of $X$ whose union is $X$, such that each connected subset of $X$ intersects only one of them.

Given a topological space $X$, we define another relation $\sim$, by requiring that $x \sim y$ if and only if there exists a path in $X$ joining $x$ and $y$. It is clear that for this relation we have for each $x \in X, x \sim x$.

For a constant map $f : [0, 1] \to X, f(t) = x$ is a path. Now suppose $x \sim y$. Then there is a path $f : [0, 1] \to X$ with $f(0) = x, f(1) = y$. Define $h : [0, 1] \to X$ by $h(t) = f(1 - t)$, it follows immediately by continuity of composition of two continuous functions that $h$ is path satisfying $h(0) = y, h(1) = x$, and consequently $y \sim x$. Finally if $x \sim y, y \sim z$, then there are paths $f : [0, 1] \to X$, and $g : [0, 1] \to X$, such that $f(0) = x, f(1) = y, g(0) = y, g(1) = z$. Define a map $h : [0, 1] \to X$ by

$$h(t) = \begin{cases} f(2t) & \text{if } t \in [0, \frac{1}{2}] \\ g(2t - 1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

which by pasting lemma is continuous and is therefore a path satisfying $h(0) = x, h(1) = z$, which proves that $x \sim z$. Consequently the relation $\sim$ is an equivalence relation and the equivalence classes
of this relation are called the path components of $X$. We have the following:

**Theorem:** The path components of a topological space $X$ are path-connected disjoint subsets of $X$ whose union is $X$, such that each path-connected subset of $X$ intersects only one of them.

Finally we define one more relation on a topological space $X$ by requiring $x \sim y$ if there is no separation $X = A \cup B$ of $X$ by disjoint open subsets such that $x \in A$ and $y \in B$. It can be shown that $\sim$ is an equivalence relation and the equivalence classes of this relation are called the quasicomponents of $X$.

**Remark:** Naturally all connected spaces have only one component the space itself. The subspace $X = (1, 2) \cup (3, 4)$ of $\mathbb{R}$ is disconnected and has components $(1, 2), (3, 4)$. These are also the path-components of $X$. The deleted comb space has one component the space itself and two path-components.

### Exercise set-3

1. Show that a component of a space is a closed set. Give an example of a topological space in which a component is not an open set.

2. If a space has finitely many components, then show that each component is an open set.

3. Show that a homeomorphism $f : X \to Y$ induces a one-to-one correspondence between components of $X$ and the components of $Y$.

4. Give example of spaces $X$ and $Y$ for which there is a one-to-one correspondence between components of $X$ and the components of $Y$ with corresponding components homeomorphic, but $X$ is not homeomorphic to $Y$.

5. For a topological space $X$ show that the component of $x \in X$ is contained in the quasicomponent of $x$. If $X$ is compact Hausdorff, show that the component of a point coincides with the quasicomponent.

6. For a topological space $X$ show that the path component of $x \in X$ is path-connected and is contained in the component of $x$.

7. Give an example of a topological space $X$ to show that a path component of $X$ need not be closed set.

8. Show that the quasicomponent of a point $x$ in a topological space $X$ is the intersection of all open-and-closed subsets of $X$ which
contain the point $x$, and that a quasicomponent of a point is a closed set.

9. Let

$$X = \{(0,0),(1,0)\} \cup \left\{\left(x,\frac{1}{n}\right) : 0 \leq x \leq 1, n \in \mathbb{Z}_+\right\}$$

be the subspace of $\mathbb{R}^2$. Find the component and quasicomponent of the point $(0,0)$.

10. Show that the subspaces $X = (0,1) \cup (2,3)$ and $Y = (0,1) \cup [2,3]$ are not homeomorphic.

11. Find the components and quasicomponents of $R_l$ ($R$ with lower limit topology).

**Local connectedness:** Note that connectedness and path-connectedness are global properties of a topological space. Now we shall introduce some local properties called the local connectedness and local path-connectedness.

**Definition:** A topological space $X$ is said to be locally connected at $x \in X$, if for each neighbourhood $U$ of $x$ there is a connected neighbourhood $V$ of $x$ contained in $U$. A topological space $X$ is said to be locally connected if it is locally connected at each of its points.

The notions connectedness and local connectedness are not related to each other: a space could possess one of these properties or both or none of these properties. Similarly we define the local path-connectedness as follows:

**Definition:** A topological space $X$ is said to be locally path-connected at $x \in X$, if for each neighbourhood $U$ of $x$ there is a path-connected neighbourhood $V$ of $x$ contained in $U$. A topological space $X$ is said to be locally path-connected if it is locally path-connected at each of its points.

**Examples:** 1) The space $\mathbb{R}$ is connected as well as locally connected.

2) The subspace $X = [1,2) \cup (2,3]$ of $\mathbb{R}$ is not connected however it is locally connected.

3) The deleted comb space is connected but not locally connected.
4) The set $Q$ of rationals as subspace of $R$ is neither connected nor locally connected.

5) Consider the subspace $Y = D \cup \{(0, \frac{1}{n}) : n \in \mathbb{Z}_+\}$ of $R^2$, where $D$ is the deleted comb space. The space $Y$ is locally connected at $(0, 0)$ but is not locally path connected at this point.

We have the following results on locally connectedness and locally path connectedness:

**Theorem 1.** A topological space $X$ is locally connected if and only if for every open set $U$ of $X$, each component of $U$ is open in $X$.

**Theorem 2.** A topological space $X$ is locally path-connected if and only if for every open set $U$ of $X$, each path-component of $U$ is open in $X$.

**Theorem 3.** If a topological space $X$ is locally path-connected, then the components and the path components of $X$ are the same.

(For proofs see Munkres p-162)

**Exercise set-4**

1. Show that if $X$ is locally connected then the quasicomponents and components of $X$ are the same.

2. If $f : X \to Y$ is continuous and $X$ is locally connected, is $f(X)$ necessarily locally connected? What if $f$ is both continuous and open?

3. Suppose $X$ is locally path-connected. Show that every connected open subset of $X$ is path connected.

4. Show that if $X$ is locally path-connected then each path component of $X$ is both open and closed.

5. Show that if $X$ is locally connected then each component of $X$ is both open and closed.

6. Show that a connected locally path-connected space is path-connected.