STRUCTURE EQUATIONS

Let $M$ be a smooth manifold and $\nabla$ be a linear connection on $M$. Let $\{e_1, \ldots, e_n\}$ be a local frame of vector field on an open subset $U$ of $M$ and $\{\omega^1, \ldots, \omega^n\}$ be the dual frame of 1-forms (that is $\omega^i(e_j) = \delta^i_j$ holds). For a smooth vector field $X \in \mathfrak{X}(M)$, we define

$$\nabla_X e_i = \sum_{j=1}^n \omega^j_i(X)e_j$$

which define smooth 1-forms $\omega^j_i : \mathfrak{X}(M) \to C^\infty(M)$ called connection forms of the connection $\nabla$. Recall that the torsion tensor field $T$ and the curvature tensor field $R$ of the connection $\nabla$ are defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

$$R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

for $X, Y, Z \in \mathfrak{X}(M)$. We shall denote by $T^i$ and $\Omega^j_i$ the components of the torsion tensor field $T$ and the curvature tensor field $R$ defined by

$$T(X, Y) = \sum_{i=1}^n T^i(X, Y)e_i \quad \text{and} \quad \Omega^j_i(X, Y) = \omega^j_i (R(X, Y)e_i), \quad X, Y \in \mathfrak{X}(M)$$

Note that $T^i(X, Y) = \omega^i(T(X, Y))$ holds. Now we prove the following

**Theorem:** On a smooth Riemannian manifold $M$ with linear connection $\nabla$ the exterior derivatives of the 1-forms $\omega^i$ and $\omega^j_i$ are given by

$$d\omega^i = \sum_{k=1}^n \omega^k \Lambda \omega^i_k + T^i$$

and

$$d\omega^j_i = \sum_{k=1}^n \omega^j_k \Lambda \omega^i_k + \Omega^j_i$$
Proof: We have using definition of $T$ and $T^i$

\[
d\omega^i(X,Y) = X(\omega^i(Y)) - Y(\omega^i(X)) - \omega^i([X,Y])
\]

\[
= X(\omega^i(Y)) - Y(\omega^i(X)) + \omega^i(T(X,Y) + \nabla_Y X - \nabla_X Y)
\]
\[
= X(\omega^i(Y)) - Y(\omega^i(X)) + \omega^i(\nabla_Y X) - \omega^i(\nabla_X Y) + T^i(X,Y)(2)
\]

Note that

\[
X = \sum_{k=1}^{n} \omega^k(X)e_k
\]
holds for each $X$, it follows from the fact that $\omega^i(e_j) = \delta^i_j$. Thus we have

\[
\nabla_Y X = \sum_{k=1}^{n} \left( Y \left( \omega^k(X) \right) e_k + \omega^k(X)\nabla_Y e_k \right)
\]
\[
= \sum_k Y \left( \omega^k(X) \right) e_k + \sum_{jk} \omega^k(X)\omega^j_k(Y)e_j
\]

where we have used (1) to substitute for $\nabla_Y e_k$. Thus from above equation we get

\[
\omega^i(\nabla_Y X) = Y(\omega^i(X)) + \sum_k \omega^k(X)\omega^j_k(Y)(3)
\]

Similarly we get

\[
\omega^j(\nabla_X Y) = X(\omega^j(Y)) + \sum_k \omega^k(Y)\omega^j_k(X)(4)
\]

Substituting equations (3) and (4) into equation (2), we get

\[
d\omega^i(X,Y) = \sum_k \omega^k(X)\omega^i_j(Y) - \sum_k \omega^k(Y)\omega^i_k(X) + T^i(X,Y)
\]
\[
= \sum_k \omega^k \Lambda \omega^i_k(X,Y) + T^i(X,Y)
\]
\[
= \left( \sum_k \omega^k \Lambda \omega^i_k + T^i \right)(X,Y)
\]

for all $X,Y \in \mathfrak{X}(M)$. This proves the first structure equation.

Now to prove the second structure equation, we have

\[
d\omega^j_i(X,Y) = X\omega^j_i(Y) - Y\omega^j_i(X) - \omega^j_i([X,Y])(5)
\]
\[ \Omega^j_i(X, Y) = \omega^j_i(R(X, Y)e_j) \]
\[ = \omega^j_i(\nabla_X \nabla_Y e_j) - \omega^j_i(\nabla_Y \nabla_X e_j) - \omega^j_i(\nabla_{[X, Y]} e_j) \]  
(6)

Since
\[ \nabla_Y e_j = \sum_k \omega^i_j(Y) e_k \]
we have
\[ \nabla_X \nabla_Y e_j = \sum_k X \left( \omega^k_j(Y) \right) e_k + \sum_{kl} \omega^k_j(Y) \omega^l_k(X) e_l \]
which gives
\[ \omega^i_j(\nabla_X \nabla_Y e_j) = X \left( \omega^i_j(Y) \right) + \sum_k \omega^k_j(Y) \omega^l_k(X) \]  
(7)

Similarly we get
\[ \omega^i_j(\nabla_Y \nabla_X e_j) = Y \left( \omega^i_j(X) \right) + \sum_k \omega^k_j(X) \omega^l_k(Y) \]  
(8)

Also we have
\[ \nabla_{[X, Y]} e_j = \sum_k \omega^i_j([X, Y]) e_k \]
which gives
\[ \omega^i_j(\nabla_{[X, Y]} e_j) = \omega^i_j([X, Y]) \]  
(9)

Using equation (7), (8) and (9) in equation (6), we get
\[ \Omega^j_i(X, Y) = X \left( \omega^j_i(Y) \right) - Y \left( \omega^j_i(X) \right) - \omega^j_i([X, Y]) + \sum_k \omega^k_j(Y) \omega^l_k(X) - \sum_k \omega^k_j(X) \omega^l_k(Y) \]
that is
\[ \Omega^j_i(X, Y) = d\omega^j_i(X, Y) - \sum_k \omega^k_j \Lambda \omega^l_k(X, Y) \]
for all \( X, Y \in \mathfrak{X}(M) \). This proves the second equation.