STRUCTURE EQUATIONS

Let M be a smooth manifold and ∇ be a linear connection on M. Let $\{e_1..., e_n\}$ be a local frame of vector field on an open subset U of M and $\{\omega^1, .., \omega^n\}$ be the dual frame of 1-fomrs (that is $\omega^i(e_j) = \delta_i^j$ holds). For a smooth vector field $X \in \mathfrak{X}(M)$, we define

$$\nabla_X e_i = \sum_{j=1}^{n} \omega_i^j(X) e_j \tag{1}$$

which define smooth 1-forms $\omega_i^j : \mathfrak{X}(M) \to C^\infty(M)$ called connection forms of the connection ∇. Recall that the torsion tensor field T and the curvature tensor field R of the connection ∇ are defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X, Y, Z \in \mathfrak{X}(M)$. We shall denote by T^i and Ω_i^j the components of the torsion tensor field T and the curvature tensor field R defined by

$$T(X,Y) = \sum_{i=1}^{n} T^i(X,Y) e_i \text{ and } \Omega_i^j(X,Y) = \omega^j(R(X,Y)e_i), \quad X, Y \in \mathfrak{X}(M)$$

Note that $T^i(X,Y) = \omega^i(T(X,Y))$ holds. Now we prove the following

Theorem: On a smooth Riemannian manifold M with linear connection ∇ the exterior derivatives of the 1-forms ω^i and ω_j^i are given by

$$d\omega^i = \sum_{k=1}^{n} \omega^k \Lambda \omega_k^i + T^i$$

and

$$d\omega_j^i = \sum_{k=1}^{n} \omega_j^k \Lambda \omega_k^i + \Omega_j^i$$

Proof: We have using definition of T and T^i

$$\begin{aligned} d\omega^i(X,Y) &= X(\omega^i(Y)) - Y(\omega^i(X)) - \omega^i([X,Y]) \\ &= X(\omega^i(Y)) - Y(\omega^i(X)) + \omega^i(T(X,Y) + \nabla_Y X - \nabla_X Y) \\ &= X(\omega^i(Y)) - Y(\omega^i(X)) + \omega^i(\nabla_Y X) - \omega^i(\nabla_X Y) + T^i(X,Y) \end{aligned} \quad (2)$$

Note that

$$X = \sum_{k=1}^{n} \omega^k(X) e_k$$

holds for each X, it follows from the fact that $\omega^i(e_j) = \delta_i^j$. Thus we have

$$\begin{aligned} \nabla_Y X &= \sum_{k=1}^{n} \left(Y\left(\omega^k(X)\right) e_k + \omega^k(X) \nabla_Y e_k \right) \\ &= \sum_k Y\left(\omega^k(X)\right) e_k + \sum_{jk} \omega^k(X) \omega_k^j(Y) e_j \end{aligned}$$

where we have used (1) to substitute for $\nabla_Y e_k$. Thus from above equation we get

$$\omega^i(\nabla_Y X) = Y\left(\omega^i(X)\right) + \sum_k \omega^k(X)\omega_k^i(Y) \quad (3)$$

Similarly we get

$$\omega^i(\nabla_X Y) = X\left(\omega^i(Y)\right) + \sum_k \omega^k(Y)\omega_k^i(X) \quad (4)$$

Substituting equations (3) and (4) into equation (2), we get

$$\begin{aligned} d\omega^i(X,Y) &= \sum_k \omega^k(X)\omega_k^i(Y) - \sum_k \omega^k(Y)\omega_k^i(X) + T^i(X,Y) \\ &= \sum_k \omega^k \Lambda \omega_k^i(X,Y) + T^i(X,Y) \\ &= \left(\sum_k \omega^k \Lambda \omega_k^i + T^i \right)(X,Y) \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$. This proves the first structure equation.

Now to prove the second structure equation, we have

$$d\omega_j^i(X,Y) = X\omega_j^i(Y) - Y\omega_j^i(X) - \omega_j^i([X,Y]) \quad (5)$$

and

$$\begin{aligned}\Omega^i_j(X,Y) &= \omega^i(R(X,Y)e_j) \\ &= \omega^i(\nabla_X\nabla_Y e_j) - \omega^i(\nabla_Y\nabla_X e_j) - \omega^i(\nabla_{[X,Y]}e_j) \qquad (6)\end{aligned}$$

Since

$$\nabla_Y e_j = \sum_k \omega^k_j(Y)e_k$$

we have

$$\nabla_X\nabla_Y e_j = \sum_k X\left(\omega^k_j(Y)\right)e_k + \sum_{kl}\omega^k_j(Y)\omega^l_k(X)e_l$$

which gives

$$\omega^i\left(\nabla_X\nabla_Y e_j\right) = X\left(\omega^i_j(Y)\right) + \sum_k \omega^k_j(Y)\omega^i_k(X) \qquad (7)$$

Similarly we get

$$\omega^i\left(\nabla_Y\nabla_X e_j\right) = Y\left(\omega^i_j(X)\right) + \sum_k \omega^k_j(X)\omega^i_k(Y) \qquad (8)$$

Also we have

$$\nabla_{[X,Y]}e_j = \sum_k \omega^k_j([X,Y])e_k$$

which gives

$$\omega^i\left(\nabla_{[X,Y]}e_j\right) = \omega^i_j([X,Y]) \qquad (9)$$

Using equation (7),(8) and (9) in equation (6), we get

$$\Omega^i_j(X,Y) = X\left(\omega^i_j(Y)\right) - Y\left(\omega^i_j(X)\right) - \omega^i_j([X,Y]) + \sum_k \omega^k_j(Y)\omega^i_k(X) - \sum_k \omega^k_j(X)\omega^i_k(Y)$$

that is

$$\Omega^i_j(X,Y) = d\omega^i_j(X,Y) - \sum_k \omega^k_j \Lambda \omega^i_k(X,Y)$$

for all $X, Y \in \mathfrak{X}(M)$. This proves the second equation.