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Conformal Riemannian Maps between Riemannian Manifolds, Their Harmonicity and Decomposition Theorems

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Abstract Riemannian maps were introduced by Fischer (Contemp. Math. 132:331–366, 1992) as a generalization isometric immersions and Riemannian submersions. He showed that such maps could be used to solve the generalized eikonal equation and to build a quantum model. On the other hand, horizontally conformal maps were defined by Fuglede (Ann. Inst. Fourier (Grenoble) 28:107–144, 1978) and Ishihara (J. Math. Kyoto Univ. 19:215–229, 1979) and these maps are useful for characterization of harmonic morphisms. Horizontally conformal maps (conformal maps) have their applications in medical imaging (brain imaging)and computer graphics. In this paper, as a generalization of Riemannian maps and horizontally conformal submersions, we introduce conformal Riemannian maps, present examples and characterizations. We show that an application of conformal Riemannian maps can be made in weakening the horizontal conformal version of Hermann's theorem obtained by Okrut (Math. Notes 66(1):94–104, 1999). We also give a geometric characterization of harmonic conformal Riemannian maps and obtain decomposition theorems by using the existence of conformal Riemannian maps.

Keywords Isometric immersion · Riemannian submersion · Horizontally conformal submersion · Riemannian map · Conformal Riemannian map

Mathematics Subject Classification (2000) 53C20 · 53C43 · 53C12

1 Introduction

The theory of smooth maps between Riemannian manifolds has been widely studied in Riemannian geometry. Such maps are useful for comparing geometric structures between two manifolds. In this point of view, isometric immersions are basic such maps between Riemannian manifolds and they are characterized by their Riemannian metrics and Jacobian matrices. More precisely, a smooth map $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ between Riemannian

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manifolds (M_1^m, g_1) and (M_2^n, g_2), where $\dim(M_1) = m$ and $\dim(M_2) = n$, is called an isometric immersion if F_* is injective and

$$g_2(F_*X, F_*Y) = g_1(X, Y) \tag{1.1}$$

for X, Y vector fields tangent to M_1^m, here F_* denotes the derivative map. It is known that the theory of isometric immersions originated from Gauss's studies on surfaces of Euclidean spaces.

On the other hand, the study of Riemannian submersions between Riemannian manifolds was initiated by O'Neill [31] and Gray [18], see also [12] and [41]. A smooth map $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is called an Riemannian submersion if F_* is onto and it satisfies (1.1) for vector fields tangent to the horizontal space $(\ker F_*)^{\perp}$. The simplest example of a Riemannian submersion is the projection of a Riemannian product manifold on one of its factors.

In 1992, Fischer introduced Riemannian maps between Riemannian manifolds in [13] as a generalization of the notions of isometric immersions and Riemannian submersions. Let $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ be a smooth map between Riemannian manifolds such that $0 < \operatorname{rank} F < \min\{m, n\}$, where $\dim M_1 = m$ and $\dim M_2 = n$. Then we denote the kernel space of F_* by $\ker F_*$ and consider the orthogonal complementary space $\mathcal{H} = (\ker F_*)^{\perp}$ to $\ker F_*$. Then the tangent bundle of M_1^m has the following decomposition

$$TM_1 = \ker F_* \oplus \mathcal{H}.$$

We denote the range of F_* by range F_* and consider the orthogonal complementary space $(\operatorname{range} F_*)^{\perp}$ to range F_* in the tangent bundle TM_2 of M_2. Since $\operatorname{rank} F < \min\{m, n\}$, we always have $(\operatorname{range} F_*)^{\perp}$ is non empty. Thus the tangent bundle TM_2 of M_2 has the following decomposition

$$TM_2 = (\operatorname{range} F_*) \oplus (\operatorname{range} F_*)^{\perp}.$$

Now, a smooth map $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is called Riemannian map at $p_1 \in M$ if the horizontal restriction $F^h_{*p_1} : (\ker F_{*p_1})^{\perp} \longrightarrow (\operatorname{range} F_{*p_1})$ is a linear isometry between the inner product spaces $((\ker F_{*p_1})^{\perp}, g_1(p_1) \mid_{(\ker F_{*p_1})^{\perp}})$ and $(\operatorname{range} F_{*p_1}, g_2(p_2) \mid_{(\operatorname{range} F_{*p_1})})$, $p_2 = F(p_1)$. Therefore Fischer stated in [13] that a Riemannian map is a map which is as isometric as it can be. In another words, F_* satisfies (1.1) for X, Y vector fields tangent to $\mathcal{H}$. It follows that isometric immersions and Riemannian submersions are particular cases of Riemannian maps with $\ker F_* = \{0\}$ and $(\operatorname{range} F_*)^{\perp} = \{0\}$. It is known that a Riemannian map is a subimmersion [13], we refer to Sect. 5 for further details.

On the other hand, a smooth map $\varphi : (M^m, g_M) \longrightarrow (N^n, g_N)$ is said to be weakly conformal at p if there is a number $\Lambda(p) = \lambda^2(p)$ such that

$$g_N(\varphi_* X, \varphi_* Y) = \lambda^2(p) g_M(X, Y) \tag{1.2}$$

for $X, Y \in T_pM$, (for more information on weakly conformal maps, we refer [3]). It is known that a smooth map $\varphi : (M^m, g_M) \longrightarrow (N^n, g_N)$ is weakly conformal at p if and only if precisely one of the following holds: (a)$\varphi_{*p} = 0$, (b)φ_{*p} is a conformal injection from T_pM into $T_{\varphi(p)}N$. A point p of type (a) in the above characterization is called a branch point of φ and a point of type (b) is called a regular point. If weakly conformal map φ has no branch points, then it is an immersion on its whole domain. Such map is called conformal immersion. We note that $\dim(M) < \dim(N)$ for a nonconstant weakly conformal map [3]. It is clear that an isometric immersion is a weakly conformal map with $\sqrt{\Lambda} = 1$.

Let (M^m, g_M) and (N^n, g_N) be Riemannian manifolds. Suppose that $\varphi : (M^m, g_M) \longrightarrow (N^n, g_N)$ is a smooth map between Riemannian manifolds and $p \in M$. Then, φ is called horizontally weakly conformal map at p if either (i) $\varphi_{*p} = 0$ or (ii) φ_{*p} maps the horizontal space $\mathcal{H}_p = (\ker(\varphi_{*p}))^{\perp}$ conformally onto $T_{\varphi(p)}N$, i.e., φ_{*p} is surjective and φ_* satisfies the equation (1.2) for X, Y vectors tangent to $\mathcal{H}_p$. If a point p is of type (i), then it is called critical point of φ. A point p of type (ii) is called regular. The number $\Lambda(p)$ is called the square dilation, it is necessarily non-negative. Its square root $\lambda(p) = \sqrt{\Lambda(p)}$ is called the dilation. A horizontally weakly conformal map $\varphi : M \longrightarrow N$ is said to be horizontally homothetic if the gradient of its dilation λ is vertical, i.e., $\mathcal{H}(\operatorname{grad} \lambda) = 0$ at regular points. If a horizontally weakly conformal map φ has no critical points, then it is called horizontally conformal submersion [3]. Thus, it follows that a Riemannian submersion is a horizontally conformal submersion with dilation identically one. We note that horizontal conformal maps were introduced independently by Fuglede [15] and Ishihara [25]. From the above discussion, one can conclude that the notion of horizontal conformal maps is a generalization of the concept of Riemannian submersions.

The concept of harmonic maps and morphisms constitutes a very useful tool for both Global Analysis and Differential Geometry. The theory of harmonic maps has been developed since 1964 [11]. This theory is still an active field in differential geometry and it has applications to many different areas of mathematics and physics. A map between Riemannian manifolds is harmonic if the divergence of its differential vanishes. Harmonic maps between Riemannian manifolds satisfy a system of quasi-linear partial differential equations. In order to have existence results one would solve PDE's on certain manifolds. On the other hand, harmonic morphisms are maps between Riemannian manifolds which preserve germs of harmonic functions i.e. these (locally) pull back real-valued harmonic functions to real-valued harmonic functions. These are characterized as harmonic maps which are horizontally (weakly) conformal. Hence, harmonic morphisms can be viewed as a sub-class of harmonic maps.

All these maps we mentioned above have their applications in many different areas. First note that immersed (or embedded) submanifolds of certain spacetime manifolds are useful tools to obtain spacetime knowledge. Indeed, the idea that the universe we live in can be represented as four dimensional submanifold embedded in a $4 + d$-dimensional spacetime manifold has attracted the attention of many physicists, more details can be found in standard books, see: [4] and [21]. Let us mention some basic notations from spacetime geometry to show how the notions of the submanifold theory have close relationship with the physical events. An n-dimensional semi-Riemannian manifold is a pair (M, g_M), where M is an n-dimensional differentiable manifold and g_M is a symmetric, non-degenerate 2-tensor field, called the semi-Riemannian metric, on M. A semi-Riemannian manifold is said to be Lorentzian if g_M has signature $(-, +, \ldots, +)$. A spacetime (M, g_M, ∇) is a connected 4-dimensional, oriented and time oriented Lorentzian manifold (M, g_M) together with the Levi-Civita connection ∇ of g_M on M. Let (M, g_M, ∇) be a spacetime manifold, then an observer in M is a future pointing timelike curve $\alpha : I \longrightarrow M$ such that $| \alpha_* | = 1$. The timelike requirement of this definition is mathematical translation of assumption that each observer travels slower than light. A reference frame Q on a spacetime M is a vector field each of whose integral curves is an observer. Let N be a 3-manifold and (M, g_M, ∇) a spacetime. Suppose that $\beta : N \longrightarrow (M, g_M)$ is a spacelike imbedding. Thus with $g_N = \beta^* g_N$, (N, g_N) is a Riemannian manifold (submanifold of M). It is known that there exists a unique symmetric 2-tensor field h on N with the following: Let $\mu : I \longrightarrow N$ be a geodesic of (N, g_N),

$\nu = \beta \circ \mu$. Then $\nabla_{\nu_*} \nu_* = h(\mu_*, \mu_*)Z$, where Z is a future pointing timelike vector field over ν which is orthogonal to $\beta(N)$. h is defined as the second fundamental form of β. Roughly speaking, h measures how much geodesics of N are curved in M. Let Z be a reference frame on M such that physically equivalent 1-form ζ obeys $\beta^*\zeta = 0$. Then $h = -\beta^*(\nabla\zeta)$. Thus h also measures how much the unit normal to N tilts as we move on around N, for details see [33].

We also note that there are many applications of Riemannian (or Semi-Riemannian) submersions, harmonic maps and conformal maps. We first review some applications of these maps taken from [12]. The standard Yang-Mills equation arises in physics when M is a Minkowski spacetime, but if one passes to the imaginary time, M becomes the Euclidean 4-space $\mathbf{R}^4$. Since in dimension 4 the Yang-Mills equation are conformally invariant, $\mathbf{R}^4$ can be replaced by $\mathbf{S}^4$. In this direction, Watson [40] gave a geometric interpretation of Yang-Mills theory on $\mathbf{S}^4$ by using Riemannian submersions which are projections. Note that a modern formulation of Yang-Mills theory requires the use of principal fibre bundles, for such bundles and the basic tools on them, see: [10].

A current approach in modern physics is the search for a theory which provides a unification of the gravity with the other fundamental forces of the nature. One of the early possibilities of such unification was suggested by Klauza and later expanded by Klein. It was shown within a 5-dimensional extension of Einstein's theory of general relativity how both gravity and electromagnetism could be treated on similar footing. Both interactions were described as part of the five dimensional metric. A natural generalization of the original Kluza-Klein idea which incorporates non-Abelian gauge fields is to consider a higher than five dimensional theory in which the gauge fields become part of the metric in the same way as the electromagnetic field did in Klauza's theory. In the modern Klauza-Klein theories one starts with the hypothesis that the spacetime has $(4+d)$-dimensions. The extra d spatial dimensions are static and curled up into a compact manifold unobservable small size, typically of order of the Plack length. In large class of physically interesting models in $(4+d)$-dimensions the compactification of the extra d-dimensions is produced spontaneously by the non-trivial vacuum configuration of an antisymmetric tensor field. It is almost common to these models to get a huge cosmological constant for the spacetime if the extra dimensions are Planck-sized. Recently, it was investigated the possibility to extend the model, assuming that the internal d-dimensional space can be larger than the manifold in which the field takes values. The general solution of the model can be expressed in terms of harmonic maps satisfying Einstein equations. It is known that a very general class of solutions is given by Riemannian submersions from the extra dimensional space onto the space in which the scalar fields take values. Spacetime with more than four dimensions has often been considered in attempt to find unified theory including gravity. The supergravity and superstring theories strongly suggest the use of a higher dimensional background.

On the other hand, in [14], Fischer proved a generalized version of the regular interval theorem of Morse theory by using Riemannian submersions. He also showed there is a close relationship between Riemannian submersion and a solution of the eikonal equation of geometrical optics for a real valued function $f : M \longrightarrow \mathbf{R}$ on a non-compact Riemannian manifold. Also recall that Hermann's theorem tells that if $f : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is a surjective submersion between Riemannian manifolds (M_1, g_1) and (M_2, g_2), M_1 is connected and (M_1, g_1) is complete, then f is a locally trivial bundle over M_2. It is known that a surjective Riemannian map on connected manifold is a surjective Riemannian submersion [22]. This result can be made in weakening the hypothesis of the above famous theorem of Hermann. Indeed, in this case we have the following: if $f : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is a surjective Riemannian map between Riemannian manifolds (M_1, g_1) and (M_2, g_2), M_1 is connected and (M_1, g_1) is complete, then f is a surjective Riemannian submersion and f is a locally trivial bundle over M_2 [13].

Let $\psi : M \to N$ be a smooth map between two Riemannian manifolds. Consider $L(g, \psi) = \rho - \gamma(\psi)$ a Lagrangian associated with the map ψ couple to gravity, where ρ is the Ricci tensor field of (M, g) and $\gamma \neq 0$ is the coupling constant. The Euler-Lagrange equations associated with $L(g, \psi) = \rho - \gamma(\psi)$ are given by

$$\rho - \frac{r}{2} g = \gamma S_\psi, \tag{1.3}$$

$$\tau(\psi) = 0, \tag{1.4}$$

where r is the scalar curvature of (M, g), S_ψ is the stress-energy tensor associated with ψ and $\tau(\psi)$ is the tension field of ψ. A horizontally conformal submersion $\psi : (M^m, g) \longrightarrow (N^n, h)$, $m > n \geq 2$, is said to be coupled to gravity if ψ and g satisfy (1.3). From (1.2) and $\operatorname{div} S_\psi = -h(\tau(\psi), \psi_*)$, it follows that S_ψ is divergence free. Any solution of (1.3) also satisfies (1.4). Thus we conclude that any horizontally conformal submersion coupled to gravity is a harmonic map (then Fuglede-Ishihara's theorem ([15] and [25]) which tells us that a smooth map is a harmonic morphism if and only if it is a harmonic horizontally weakly conformal map implies that it is a harmonic morphism). In [28], Mustafa gives some necessary conditions to construct horizontally conformal submersion coupled to gravity on compact Riemannian manifolds.

Recently, conformal structures and harmonic maps have been used in geometric modelling and computer vision which are widely used in medical imaging [27, 36, 38]. Several powerful brain mapping techniques have emerged since the mid 1990's. Many rely on computational anatomy, a mathematical brain modelling approach in which brain surfaces and subvolumes are viewed as complex geometrical patterns and are modelled as $3D$ continuous mesh models (or deformable shapes) that can be averaged and combined across subjects [2]. To compare and integrate brain data, data from multiple subjects are typically mapped into a canonical space surface based approaches often map cortical surface data to parameter domain such as sphere or $2D$ plane providing a common coordinate system for data integration. One method to do this is to conformally map cortical surfaces to the sphere.The cortical surface of the brain is topologically equivalent to a 2-sphere (i.e. a closed, simply-connected surface with no holes or handles and has an Euler characteristic of 2) [24]. For genus zero surfaces (Riemannian) harmonic maps and conformally maps are equivalent and any genus zero Riemannian surface can be mapped conformally to a sphere. Since the cortical surface of the brain surface is a genus zero surface, conformally mapping offers a convenient method to parametrize cortical surfaces without angular distortion generating an orthogonal grid on the cortex that locally preserves the metric [37]. In [20], the authors propose a method to find unique conformal mapping between any two genus zero manifolds by minimizing the harmonic energy of the map. They demonstrate this method by conformally mapping a cortical surface to a sphere. In [39], the authors used the Ricci-flow method to compute a conformal mapping between cortical surfaces and a multi-hole surface. Then they computed a direct cortical surface correspondence by computing a constrained harmonic map on the parameter domain. We also note that it is known that there is close relationship between finding harmonic map from 3-manifold to $3D$ sphere and volume mapping problem [19, 42].

On the other hand, in [26] the author gives a formulation of semi-conformal (horizontally conformal) maps in terms of jets which are important notions in synthetic differential geometry. He shows that a map F is a semi-conformal harmonic map if and only if it pulls back harmonic 2-jets to harmonic 2-jets.

In this paper, as a generalization of conformal immersions, horizontally conformal submersions and Riemannian maps, we introduce conformal Riemannian maps. In Sect. 2, we give some basic knowledge for distributions and harmonic maps. In Sect. 3, we define conformal Riemannian map and give examples. Using the conformal Riemannian notion, we obtain a new version of Okrut's theorem which is a generalization of Hermann's theorem to horizontally conformal submersion. We also obtain two characterizations. In Sect. 4, we study harmonicity of conformal Riemannian maps and obtain conditions for conformal Riemannian maps to be harmonic.

An important initial step in understanding a complicated mathematical object is to decompose it into simpler irreducible components. In differential geometry, a fundamental result in this direction is the decomposition theorem of de Rham, which gives necessary and sufficient conditions for a Riemannian manifold to split, both locally and globally, into a Riemannian product of Riemannian manifolds [9]. A generalization of product Riemannian manifolds, warped products were introduced in [5] as a tool to construct Riemannian manifolds with non positive curvature. It is important to note that warped products comprise a wide variety of exact solutions to Einstein's field equations: Bertotti-Robinson, Robertson-Walker, Schwarzschild, Reissner-Nordstrom, de Sitter, etc. On the other hand, twisted products were introduced in [6] as a more general structure, since they include a wider variety of metrics than warped products. In [23] a decomposition theorem was given by Hiepko for Riemannian manifolds that split as a warped product of Riemannian manifolds. In [32] the authors proved a type of de Rham decomposition theorem for pseudo-Riemannian manifolds that split as a twisted product of Riemannian manifolds. In the last section of this paper, by using the existence of conformal Riemannian maps between Riemannian manifolds, we give a decomposition theorem that splits the domain manifold as a twisted product of Riemannian manifolds. We also give another decomposition theorem which shows the connection between subimmersions, horizontally conformal submersions, isometric immersions and conformal Riemannian maps.

2 Distributions and Harmonic Maps

In this section we recall some basic materials from [3] to fix our notations and terminology of this paper. Let (M^m, g_M) be a Riemannian manifold and $\mathcal{V}$ be a q-dimensional distribution on M. Denote its orthogonal distribution $\mathcal{V}^\perp$ by $\mathcal{H}$. Then, we have

$$TM = \mathcal{V} \oplus \mathcal{H}. \tag{2.1}$$

$\mathcal{V}$ is called the vertical distribution and $\mathcal{H}$ is called the horizontal distribution. We use the same letters to denote the orthogonal projections onto these distributions.

By the unsymmetrized second fundamental form of $\mathcal{V}$, we mean the tensor field $A^{\mathcal{V}}$ defined by

$$A^{\mathcal{V}}_E F = \mathcal{H}(\nabla_{\mathcal{V}E} \mathcal{V}F), \quad E, F \in \Gamma(TM), \tag{2.2}$$

where ∇ is the Levi-Civita connection on M. The symmetrized second fundamental form $B^{\mathcal{V}}$ of $\mathcal{V}$ is given by

$$B^{\mathcal{V}}(E,F) = \frac{1}{2}\{A^{\mathcal{V}}_E F + A^{\mathcal{V}}_F E\} = \frac{1}{2}\{\mathcal{H}(\nabla_{\mathcal{V}E}\mathcal{V}F) + \mathcal{H}(\nabla_{\mathcal{V}F}\mathcal{V}E)\} \tag{2.3}$$

for any $E, F \in \Gamma(TM)$. The integrability tensor of $\mathcal{V}$ is the tensor field $I^{\mathcal{V}}$ given by

$$I^{\mathcal{V}}(E, F) = A^{\mathcal{V}}_E F - A^{\mathcal{V}}_F E - \mathcal{H}([\mathcal{V}E, \mathcal{V}F]). \tag{2.4}$$

Moreover, the mean curvature of $\mathcal{V}$ is defined by

$$\mu^{\mathcal{V}} = \frac{1}{q} \operatorname{trace} B^{\mathcal{V}} = \frac{1}{q} \sum_{i=1}^{q} \mathcal{H}(\nabla_{e_r} e_r), \tag{2.5}$$

where $\{e_1, \dots, e_q\}$ is a local frame of $\mathcal{V}$. By reversing the roles of $\mathcal{V}$, $\mathcal{H}$, $B^{\mathcal{H}}$, $A^{\mathcal{H}}$ and $I^{\mathcal{H}}$ can be defined similarly. For instance, $B^{\mathcal{H}}$ is defined by

$$B^{\mathcal{H}}(E, F) = \frac{1}{2}\{\mathcal{V}(\nabla_{\mathcal{H}E}\mathcal{H}F) + \mathcal{V}(\nabla_{\mathcal{H}F}\mathcal{H}E)\} \tag{2.6}$$

and, hence we have

$$\mu^{\mathcal{H}} = \frac{1}{m-q} \operatorname{trace} B^{\mathcal{H}} = \frac{1}{m-q} \sum_{s=1}^{m-q} \mathcal{V}(\nabla_{E_s} E_s), \tag{2.7}$$

where $E_1, \dots, E_{m-q}$ is a local frame of $\mathcal{H}$. A distribution $\mathcal{D}$ on M is said to be minimal if, for each $x \in M$, the mean curvature vanishes.

Let (M^m, g_M) and (N^n, g_N) be Riemannian manifolds and suppose that $\varphi : (M^m, g_M) \longrightarrow (N^n, g_N)$ is a smooth mapping between them. Then the differential φ_* of φ can be viewed a section of the bundle $\operatorname{Hom}(TM, \varphi^{-1}TN) \longrightarrow M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibres $(\varphi^{-1}TN)_p = T_{\varphi(p)}N$, $p \in M$. $\operatorname{Hom}(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X, Y) = \nabla^{\varphi}_X \varphi_*(Y) - \varphi_*(\nabla^M_X Y) \tag{2.8}$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric. A smooth map $\varphi : (M^m, g_M) \longrightarrow (N^n, g_N)$ is said to be harmonic if $\operatorname{trace} \nabla\varphi_* = 0$. On the other hand, the tension field of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = \operatorname{div} \varphi_* = \sum_{i=1}^{m} (\nabla\varphi_*)(e_i, e_i), \tag{2.9}$$

where $\{e_1, \dots, e_m\}$ is the orthonormal frame on M. Then it follows that φ is harmonic if and only if $\tau(\varphi) = 0$.

Remark 2.1 As models for physical theories, harmonic maps lead to non-linear field equations, which in many respects bear a resemblance to Yang-Mills equations and to the Einstein equations for gravitation. Harmonic maps are also linked to the non-linear sigma models, first introduced by Schwinger [34], in the fifties, to describe massive, strongly interacting particles. Recent developments in non-linear sigma models include interaction with gravity, inflationary cosmological models, Kaluza-Klein theories. The supersymmetric version of this non-linear σ-problem was obtained an extension of the concept of harmonic maps, the so-called Dirac-harmonic maps that couple the map with a nonlinear spinor field while preserving the essential structural properties of harmonic maps, [7, 8]. Harmonic maps are also closely related to holomorphic maps in several complex variables, to the theory of stochastic processes and to the theory of liquid crystals in material science.

3 Conformal Riemannian Maps

In this section, as a generalization of Riemannian maps, we introduce conformal Riemannian maps, give some examples and two characterizations. We also show the relationship between conformal Riemannian maps and Hermann's theorem.

Definition 1 Let (M_1^m, g_1) and (M_2^n, g_2) be Riemannian manifolds and $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ a smooth map between them. Then we say that F is a conformal Riemannian map at $p_1 \in M_1$ if $0 < \operatorname{rank} F_{*p_1} \leq \min\{m, n\}$ and F_{*p_1} maps the horizontal space $\mathcal{H}(p_1) = (\ker(F_{*p_1}))^{\perp}$ conformally onto range(F_{*p_1}), i.e., there exists a number $\lambda^2(p_1) \neq 0$ such that

$$g_2(F_{*p_1}X, F_{*p_1}Y) = \lambda^2(p_1) g_1(X, Y) \tag{3.1}$$

for $X, Y \in \mathcal{H}(p_1)$. Also F is called conformal Riemannian if F is conformal Riemannian at each $p_1 \in M_1$.

We give some examples of conformal Riemannian maps.

Example 1 Let $I : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ be an isometric immersion between Riemannian manifolds. Then I is a conformally Riemannian map with $\lambda = 1$ and $\ker F_* = \{0\}$.

Example 2 Let $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ be a Riemannian submersion between Riemannian manifolds. Then F is a conformally Riemannian map with $\lambda = 1$ and $(\operatorname{range} F_*)^{\perp} = \{0\}$.

Example 3 A Riemannian map is a conformal Riemannian map with $\lambda = 1$.

Example 4 A conformal immersion is a conformal Riemannian map with $\ker F_* = \{0\}$.

Example 5 A horizontally conformal submersion is a conformal Riemannian map with $(\operatorname{range} F_*)^{\perp} = \{0\}$.

We say that a conformal Riemannian map $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is proper if $0 < \operatorname{rank} F < \min\{m, n\}$ and $\lambda \neq 1$. Here we give an example of a proper conformal Riemannian map.

Example 6 Consider a map $F : \mathbf{R}^4 \longrightarrow \mathbf{R}^5$ defined by $F(x_1, x_2, x_3, x_4) = (e^{x_3} \sin x_4, 0, 0, e^{x_3} \cos x_4)$. Then we have

$$\ker F_* = \operatorname{span}\{Z_1 = \partial x_1, Z_2 = \partial x_2\}$$

and

$$(\ker F_*)^{\perp} = \operatorname{span}\{Z_3 = \partial x_3, Z_4 = \partial x_4\}.$$

Then it follows that $\operatorname{rank} F = 2$. Also by direct computations, we get

$$F_* Z_3 = e^{x_3} \sin x_4 \partial y_1 + e^{x_3} \cos x_4 \partial y_4, \; F_* Z_4 = e^{x_3} \cos x_4 \partial y_1 - e^{x_3} \sin x_4 \partial y_4.$$

Hence, we have

$$g_2(F_* Z_3, F_* Z_3) = (e^{x_3})^2 g_1(Z_3, Z_3), \; g_2(F_* Z_4, F_* Z_4) = (e^{x_3})^2 g_1(Z_4, Z_4),$$

where g_2 and g_1 denote the standard metrics (inner products) of $\mathbf{R}^5$ and $\mathbf{R}^4$, respectively. Thus, F is a conformal Riemannian map with $\lambda = e^{x_3}$.

As for horizontally conformal maps, we say that a conformal Riemannian map is horizontally homothetic if the gradient of its dilation λ is vertical, i.e., $\mathcal{H}(\operatorname{grad}\lambda) = 0$. A generalization of Hermann's theorem was given in [30] by using horizontally conformal submersion as follow: if $f : E \longrightarrow M$ is a horizontal homothetic submersion of a complete Riemannian manifold E, then f is a projection of a locally trivial bundle f whose base space is also a complete Riemannian manifold. It is obvious that a surjective conformal Riemannian map $F : M_1 \longrightarrow M_2$ (with complete M_1) is a horizontal conformal submersion on a connected Riemannian manifold, proof is same with Theorem 1.3 of [13] or Theorem 4.3.1 of [16]. This result can be made in weakening the hypothesis of the above theorem of Okrut. In this case we have the following:

Theorem 3.1 *If $f : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is a surjective homothetic conformal Riemannian map between Riemannian manifolds (M_1^m, g_1) and (M_2^n, g_2), M_1 is connected and (M_1, g_1) is complete, then f is a projection of a locally trivial bundle f whose base space is also a complete Riemannian manifold.*

For any proper conformal Riemannian map $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ between Riemannian manifolds, the restriction of the differential F_{*p_1} to the horizontal space $(\ker F_{*p_1})^\perp$ maps that space isomorphically onto $(\operatorname{range} F_{*p_1})$. Denote its inverse by $\hat{}$, then any vector $Z \in (\operatorname{range} F_{*p_1})$, the vector $\hat{Z}$ is called the horizontal lift of Z. Now we recall the notion of the adjoint map of a linear map which will be useful for the rest of this paper. Let $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ be a smooth map between Riemannian manifolds (M_1^m, g_1) and (M_2^n, g_2). We denote the adjoint of F_* by ${}^*F_*$ which is defined by $g_2(F_*X_1, X_2) = g_1(X_1, {}^*F_*X_2)$ for $X_1 \in T_{p_1}M_1$ and $X_2 \in T_{F(p_1)}M_2$.

Theorem 3.2 *Let $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ be a proper conformal Riemannian map between Riemannian manifolds. Then, we have*

$$\|F_*\|^2(p_1) = \lambda^2(p_1)\,\operatorname{rank} F$$

at each $p_1 \in M_1$.

Proof Let ${}^*F_{*p_1}$ be the adjoint of F_{*p_1} and define a linear transformation $G : (\mathcal{H}(p_1), g_{1p_1\mathcal{H}(p_1)}) \longrightarrow (\mathcal{H}(p_1), g_{1p_1\mathcal{H}(p_1)})$ by $G_{p_1} = {}^*F_{*p_1} \circ F_{*p_1}$. Then for $X_1, Y_1 \in \mathcal{H}(p_1)$, we get

$$g_{1p_1}(G_{p_1}X_1, Y_1) = g_{1p_1}({}^*F_{*p_1} \circ F_{*p_1}X_1, Y_1) = g_{2F(p_1)}(F_{*p_1}X_1, F_{*p_1}Y_1).$$

Since F is proper conformal Riemannian, we have

$$g_{1p_1}(G_{p_1}X_1, Y_1) = \lambda^2(p_1) g_{1p_1}(X_1, Y_1).$$

Hence, it follows that $G_{p_1}X_1 = \lambda^2(p_1)X_1$. Thus, we obtain

$$\|F_*\|^2(p_1) = \sum_{i=1}^{n} g_{2F(p_1)}(F_{*p_1}e_i, F_{*p_1}e_i) = \lambda^2(p_1)\sum_{i=1}^{n_1} g_{1p_1}(e_i, e_i),$$

where $\{e_i\}, i \in \{1, \ldots, n_1 = \dim(\mathcal{H})\}$ is an orthonormal basis of $(\ker F_*)^\perp$. Then we have

$$\|F_*\|^2(p_1) = \lambda^2(p_1)\dim(\mathcal{H}) = \lambda^2(p_1)\,\mathrm{rank}\,F.$$

Thus proof is complete. □

Let $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ be a smooth map between Riemannian manifolds. Define linear transformations

$$\mathcal{P}_{p_1}: \quad T_{p_1}M_1 \longrightarrow T_{p_1}M_1, \qquad \mathcal{P}_{p_1} = {}^*F_{*p_1} \circ F_{*p_1},$$
$$\mathcal{Q}_{p_1}: \quad T_{p_2}M_2 \longrightarrow T_{p_2}M_2, \qquad \mathcal{Q}_{p_1} = F_{*p_1} \circ {}^* F_{*p_1},$$

where $p_2 = F(p_1)$. Using these linear transformations, we obtain the following characterizations of conformal Riemannian maps.

Theorem 3.3 *Let $(M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ be a smooth map between Riemannian manifolds (M_1^m, g_1) and (M_2, g_2). Then F is a conformal Riemannian map if and only if, for all $p_1 \in M_1$, there exists a smooth function λ such that*

$$\mathcal{Q}_{p_1} \circ \mathcal{Q}_{p_1} = \lambda^2 \mathcal{Q}_{p_1}.$$

Proof F is conformal Riemannian if and only if

$$g_{1p_1}(X_1, Y_1) = \frac{1}{\lambda^2(p_1)} g_{2p_2}(F_{*p_1}X_1, F_{*p_1}Y_1).$$

Since range* $F_{*p_1} = \mathcal{H}(p_1)$, this holds if and only if, for $X_2, Y_2 \in T_{p_2}M_2$

$$g_{1p_1}({}^*F_{*p_1}X_2, {}^*F_{*p_1}Y_2) = \frac{1}{\lambda^2(p_1)} g_{2p_2}(F_{*p_1} \circ^* F_{*p_1}X_2, F_{*p_1} \circ^* F_{*p_1}Y_2)$$

which is equivalent to

$$g_{2p_2}(X_2, F_{*p_1} \circ^* F_{*p_1}Y_2) = \frac{1}{\lambda^2(p_1)} g_{1p_1}({}^*F_{*p_1}X_2, {}^*F_{*p_1} \circ F_{*p_1} \circ^* F_{*p_1}Y_2)$$

that is,

$$g_{2p_2}(X_2, F_{*p_1} \circ^* F_{*p_1}Y_2) = \frac{1}{\lambda^2(p_1)} g_{2p_2}(X_2, F_{*p_1} \circ^* F_{*p_1} \circ F_{*p_1} \circ^* F_{*p_1}Y_2).$$

Since $\mathcal{Q}_{p_1} = F_{*p_1} \circ^* F_{*p_1}$, we get

$$g_{2p_2}(X_2, \mathcal{Q}_{p_1}Y_2) = \frac{1}{\lambda^2(p_1)} g_{2p_2}(X_2, (\mathcal{Q}_{p_1} \circ \mathcal{Q}_{p_1}Y_2).$$

Then, since the above equation holds for all $X_2, Y_2 \in T_{p_2}M_2$, Riemannian metric g_2 implies that $\mathcal{Q}_{p_1} \circ \mathcal{Q}_{p_1} = \lambda^2(p_1)\mathcal{Q}_{p_1}$. □

Theorem 3.4 *Let $(M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ be a smooth map between Riemannian manifolds (M_1^m, g_1) and (M_2^n, g_2). Then F is a conformal Riemannian map if and only if, for all $p_1 \in M_1$, there exists a smooth function λ such that*

$$\mathcal{P}_{p_1} \circ \mathcal{P}_{p_1} = \lambda^2 \mathcal{P}_{p_1}.$$

Proof Again F is conformal Riemannian if and only if there exists a positive function λ on M_1 such that

$$g_{2p_2}(F_{*p_1}X_1, F_{*p_1}Y_1) = \lambda^2(p_1)g_{1p_1}(X_1, Y_1)$$

for $X_1, Y_1 \in T_{p_1}M_1$. Since $g_{2p_2}(F_{*p_1}X_1, F_{*p_1}Y_1) = g_{1p_1}(X_1, {}^*F_{*p_1} \circ F_{*p_1}Y_1)$, the above statement is equivalent to

$$g_{2p_2}(F_{*p_1}X_1, F_{*p_1} \circ {}^*F_{*p_1} \circ F_{*p_1}Y_1) = \lambda^2(p_1)g_{1p_1}(X_1, {}^*F_{*p_1} \circ F_{*p_1}Y_1)$$

which is equivalent to

$$g_{1p_1}(X_1, {}^*F_{*p_1} \circ F_{*p_1} \circ {}^*F_{*p_1} \circ F_{*p_1}Y_1) = \lambda^2(p_1)g_{1p_1}(X_1, {}^*F_{*p_1} \circ F_{*p_1}Y_1).$$

Then, since ${}^*F_{*p_1} \circ F_{*p_1} = \mathcal{P}_{p_1}$, we conclude that F is conformal Riemannian map if and only if

$$g_{1p_1}(X_1, (\mathcal{P}_{p_1} \circ \mathcal{P}_{p_1}Y_1) = \lambda^2(p_1)g_{1p_1}(X_1, \mathcal{P}_{p_1}Y_1).$$

Then Riemannian metric g_1 and positive function λ^2 give the assertion of theorem. □

4 Harmonicity of Conformal Riemannian Maps

In this section we are going to study harmonicity of conformal Riemannian maps. First we give the following lemmas which will be useful for the proof of the theorem of this section.

Lemma 4.1 *Let $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ be a proper conformal Riemannian map. Then we have*

$$\sum_{a=1}^{n_1} g_2((\nabla F_*)(X, Y), \bar{Z}_a)\bar{Z}_a = X(\ln\lambda)\bar{Y} + Y(\ln\lambda)\bar{X}$$
$$- g_1(X, Y)F_*(\mathrm{grad}(\ln\lambda)), \quad \mathrm{rank}\, F_* = n_1, \qquad (4.1)$$

where $\{\bar{Z}_a\}$ is an orthonormal basis of range F_*, $\bar{X}, \bar{Y}$ *are vector fields tangent to* range F_* *and X, Y, their horizontal lifts on* $(\ker F_*)^{\perp}$.

Proof Let $\{\bar{Z}_a\}$ be an orthonormal basis of range F_*, lift of each $\bar{Z}_a$ to a horizontal vector field Z_a on M_1, then $\{\lambda Z_a\}$ is an orthonormal basis for the horizontal distribution of M_1. Let $\bar{X}$ and $\bar{Y}$be vector fields tangent to range F_* and X, Y their horizontal lifts to M_1. Then we can write

$$\mathcal{H}(\nabla_X^{M_1}Y) = \sum_{a=1}^{n_1} g_1(\nabla_X^{M_1}Y, \lambda Z_a)\lambda Z_a = \lambda^2 \sum_{a=1}^{n_1} g_1(\nabla_X^{M_1}Y, Z_a)Z_a,$$

where ∇^{M_1} is the Levi-Civita connection of M_1^m. From Kozsul identity for ∇^{M_1}, we have

$$\mathcal{H}(\nabla_X^{M_1}Y) = \frac{\lambda^2}{2}\sum_{a=1}^{n_1}\{Xg_1(Y, Z_a) + Yg_1(Z_a, X) - Z_a g_1(X, Y)$$
$$- g_1(X, [Y, Z_a]) - g_1(Y, [X, Z_a]) + g_1(Z_a, [X, Y])\}.$$

Now, using $g_1(X,Y)=\frac{1}{\lambda^2}g_2(\bar{X},\bar{Y})$, we get

$$\mathcal{H}(\nabla^{M_1}_X Y)=\frac{\lambda^2}{2}\sum_{a=1}^{n_1}\left\{X\left(\frac{1}{\lambda^2}g_2(\bar{Y},\bar{Z}_a)\right)+Y\left(\frac{1}{\lambda^2}g_2(\bar{Z}_a,\bar{X})\right)-Z_a\left(\frac{1}{\lambda^2}g_2(\bar{X},\bar{Y})\right)\right.$$
$$\left.-\frac{1}{\lambda^2}g_2(\bar{X},[\bar{Y},\bar{Z}_a])-\frac{1}{\lambda^2}g_2(\bar{Y},[\bar{X},\bar{Z}_a])+\frac{1}{\lambda^2}g_2(\bar{Z}_a,[\bar{X},\bar{Y}])\right\}.$$

Hence, we derive

$$\mathcal{H}(\nabla^{M_1}_X Y)=\sum_{a=1}^{n_1}\{-X(\ln\lambda)g_2(\bar{Y},\bar{Z}_a)-Y(\ln\lambda)g_2(\bar{Z}_a,\bar{X})$$
$$+Z_a(\ln\lambda)g_2(\bar{X},\bar{Y})+g_2(\nabla^{M_2}_{\bar{X}}\bar{Y},\bar{Z}_a)\}Z_a.$$

Then applying F_*, we get

$$F_*(\mathcal{H}(\nabla^{M_1}_X Y))=\sum_{a=1}^{n_1}\{-X(\ln\lambda)g_2(\bar{Y},\bar{Z}_a)-Y(\ln\lambda)g_2(\bar{Z}_a,\bar{X})$$
$$+Z_a(\ln\lambda)g_2(\bar{X},\bar{Y})+g_2(\nabla^{M_2}_{\bar{X}}\bar{Y},\bar{Z}_a)\}\bar{Z}_a.$$

Hence, we obtain

$$\sum_{a=1}^{n_1}g_2(F_*(\nabla^{M_1}_X Y),\bar{Z}_a)\bar{Z}_a=-X(\ln\lambda)\bar{Y}-Y(\ln\lambda)\bar{X}$$
$$+\sum_{a=1}^{n_1}Z_a(\ln\lambda)g_2(\bar{X},\bar{Y})\bar{Z}_a+g_2(\nabla^{M_2}_{\bar{X}}\bar{Y},\bar{Z}_a)\bar{Z}_a.$$

Thus, we arrive at

$$\sum_{a=1}^{n_1}g_2((\nabla F_*)(X,Y),\bar{Z}_a)\bar{Z}_a=X(\ln\lambda)\bar{Y}+Y(\ln\lambda)\bar{X}$$
$$-\sum_{a=1}^{n_1}Z_a(\ln\lambda)g_2(\bar{X},\bar{Y})\bar{Z}_a. \tag{4.2}$$

Here

$$g_2(\bar{X},\bar{Y})g_1(\mathrm{grad}(\ln\lambda)_{\mathcal{H}},Z_a)=g_1(X,Y)g_2(F_*(\mathrm{grad}(\ln\lambda)_{\mathcal{H}}),F_*Z_a). \tag{4.3}$$

Using (4.3) in (4.2), we obtain

$$\sum_{a=1}^{n_1}g_2((\nabla F_*)(X,Y),\bar{Z}_a)\bar{Z}_a=X(\ln\lambda)\bar{Y}+Y(\ln\lambda)\bar{X}$$
$$-\sum_{a=1}^{n_1}g_1(X,Y)g_2(F_*(\mathrm{grad}(\ln\lambda)_{\mathcal{H}}),\bar{Z}_a)\bar{Z}_a.$$

Hence, we get

$$\sum_{a=1}^{n_1} g_2((\nabla F_*)(X,Y),\bar{Z}_a)\bar{Z}_a = X(\ln\lambda)\bar{Y} + Y(\ln\lambda)\bar{X} - g_1(X,Y)F_*(\mathrm{grad}(\ln\lambda)).$$

□

Lemma 4.2 *Let $F:(M_1^m,g_1)\longrightarrow(M_2^n,g_2)$ be a proper conformal Riemannian map between Riemannian manifolds. Then the tension field τ of F is*

$$\tau = -(m-n_1)F_*(\mu^{\ker F_*}) + (2-n_1)F_*(\mathrm{grad}(\ln\lambda)) + n_1\mu^{\mathrm{range}\,F_*},$$

where $\mu^{\ker F_*}$ and $\mu^{\mathrm{range}\,F_*}$ are the mean curvature vector fields of the distributions of $\ker F_*$ *and* $\mathrm{range}\,F_*$, *respectively.*

Proof Let $\{e_1,\dots,e_{n_1},e_{n_1+1},\dots,e_m\}$ be an orthonormal basis of $\Gamma(TM_1)$ such that $\{e_1,\dots,e_{n_1}\}$ is an orthonormal basis of $(\ker F_*)^\perp$ and $\{e_{n_1+1},\dots,e_m\}$ is an orthonormal basis of $\ker F_*$. Then the trace of the second fundamental form (restriction to $(\ker F_*)^\perp\times(\ker F_*)^\perp$) is given by

$$\mathrm{trace}^{(\ker F_*)^\perp}\nabla F_* = \sum_{a=1}^{n_1}(\nabla F_*)(e_a,e_a) = \sum_{a=1}^{n_1}\sum_{j=1}^{n} g_2((\nabla F_*)(e_a,e_a),\breve{Z}_j)\breve{Z}_j$$

where $\{\breve{Z}_j\}$ is an orthonormal basis of $\Gamma(TM_2)$. Considering the decomposition of TM_2, we have

$$\mathrm{trace}^{(\ker F_*)^\perp}\nabla F_* = \sum_{a=1}^{n_1}\left\{\sum_{i=1}^{n_1} g_2((\nabla F_*)(e_a,e_a),\bar{Z}_i)\bar{Z}_i + \sum_{s=1}^{n_2} g_2((\nabla F_*)(e_a,e_a),\tilde{Z}_s)\tilde{Z}_s\right\},$$

where $\{\tilde{Z}_s\}$ is an orthonormal basis of $\Gamma((\mathrm{range}\,F_*)^\perp)$. Then from (4.1), we get

$$\mathrm{trace}^{(\ker F_*)^\perp}\nabla F_* = (2-n_1)F_*(\mathrm{grad}(\ln\lambda)) + \sum_{a=1}^{n_1}\sum_{s=1}^{n_2} g_2(\nabla^{M_2}_{F_*e_a}F_*e_a,\tilde{Z}_s)\tilde{Z}_s.$$

Using (2.5) and (2.7), we obtain

$$\mathrm{trace}^{(\ker F_*)^\perp}\nabla F_* = (2-n_1)F_*(\mathrm{grad}(\ln\lambda)) + n_1\sum_{s=1}^{n_2} g_2(\mu^{\mathrm{range}\,F_*},\tilde{Z}_s)\tilde{Z}_s.$$

Hence, we derive

$$\mathrm{trace}^{(\ker F_*)^\perp}\nabla F_* = (2-n_1)F_*(\mathrm{grad}(\ln\lambda)) + n_1\mu^{\mathrm{range}\,F_*}. \tag{4.4}$$

In a similar way, we have

$$\mathrm{trace}^{\ker F_*}\nabla F_* = \sum_{r=n_1+1}^{m}(\nabla F_*)(e_r,e_r) = -\sum_{r=n_1+1}^{m}F_*(\nabla_{e_r}e_r).$$

Hence, we obtain

$$\mathrm{trace}^{\ker F_*}\nabla F_* = -(m-n_1)F_*(\mu^{\ker F_*}). \tag{4.5}$$

Then proof follows from (4.4) and (4.5). □

From Lemma 4.2, we have the following theorem.

Theorem 4.1 *Suppose $F:(M_1^m,g_1)\longrightarrow(M_2^n,g_2)$ is a non-constant proper conformal Riemannian map between Riemannian manifolds. If* rank $F_*=n_1\neq 2$, *then any three conditions below imply the fourth*:

(i) *F is harmonic*,
(ii) *F is horizontally homothetic Riemannian map*,
(iii) *The distribution* $\ker F_*$ *is minimal*,
(iv) *The distribution* range F_* *is minimal*.

If $n_1=2$, then F is harmonic if and only if the distributions $\ker F_*$ *and* range F_* *are minimal*.

Then from Theorem 4.1, if F is a horizontally homothetic conformal Riemannian map, we have the following result.

Corollary 4.1 *Suppose $F:(M_1^m,g_1)\longrightarrow(M_2^n,g_2)$ is a non-constant horizontally homothetic proper conformal Riemannian map between Riemannian manifolds. Then any two conditions below imply the third*:

(i) *F is harmonic*,
(ii) *The distribution* $\ker F_*$ *is minimal*,
(iii) *The distribution* range F_* *is minimal*.

It is obvious that a Riemannian map is a horizontally homothetic conformal Riemannian map, thus we can state the following.

Corollary 4.2 *Suppose $F:(M_1^m,g_1)\longrightarrow(M_2^n,g_2)$ is a Riemannian map between Riemannian manifolds. Then any two conditions below imply the third*:

(i) *F is harmonic*,
(ii) *The distribution* $\ker F_*$ *is minimal*,
(iii) *the distribution* range F_* *is minimal*.

Remark 4.1 We note that for any C^2 real valued function f defined on an open subset U of a Riemannian manifold M, the equation $\triangle f=0$ is called Laplace's equation and solutions are called harmonic functions on U. Let $F:M\longrightarrow N$ be a smooth map between Riemannian manifolds. Then F is called harmonic morphism if, for every harmonic function $f:V\longrightarrow\mathbf{R}$ defined an open subset V of N with $F^{-1}(V)$ non-empty, the composition $f\circ F$ is harmonic on $F^{-1}(V)$. A smooth map $F:M\longrightarrow N$ between Riemannian manifolds is a harmonic morphism if and only if F is both harmonic and horizontally weakly conformal [15] and [25]. In this respect, a harmonic conformal Riemannian map is a good candidate for harmonic morphism.

5 Decomposition Theorems

In this section we give a characterization of twisted product manifold by the existence of conformal Riemannian maps and then we obtain another decomposition theorem which gives the relation between conformal Riemannian maps, isometric immersions, subimmersions and horizontal conformal submersions. First, let us recall the notion of twisted product. Let (M^m, g_M) and (N^n, g_N) be Riemannian manifolds of dimensions m and n, respectively. Let $P_1 : M \times N \longrightarrow M$ and $P_2 : M \times N \longrightarrow N$ be the canonical projections. Suppose that $f : M \times N \longrightarrow (0, \infty)$ be a smooth function. Then the twisted product of (M^m, g_M) and (N^n, g_N) with twisting function f is defined to be the product manifold $\bar{M} = M \times N$ with metric tensor $\bar{g} = g_M \oplus f^2 g_N$ given by

$$\bar{g} = P_1^* g_M + f^2 P_2^* g_N. \tag{5.1}$$

We denote this twisted product manifold $(\bar{M}, \bar{g})$ by $M \times_f N$. A local characterization of a twisted product can be stated in terms of the distributions D_M and D_N of the product manifold $\bar{M} = M \times N$ as follows. Let $\bar{g}$ be a Riemannian metric tensor on the manifold $\bar{M} = M \times N$ and assume that the distributions D_M and D_N intersect perpendicularly everywhere. Then $\bar{g}$ is the metric tensor of a twisted product $M \times_f N$ if and only if D_M is a totally geodesic foliation and D_N is a totally umbilical foliation [32].

We also recall that a map $F : (M^m, g_M) \longrightarrow (N^n, g_N)$ between Riemannian manifolds is called umbilical [29, 35] if

$$\nabla F_* = \frac{1}{m} g_M \otimes \tau. \tag{5.2}$$

Theorem 5.1 *Let $F : (M^m, g_M) \longrightarrow (N^n, g_N)$ be an umbilical conformal Riemannian map between Riemannian manifolds. Then (M^m, g_M) is locally a twisted product manifold $M_{(\ker F_*)^\perp} \times_f M_{\ker F_*}$, where $M_{(\ker F_*)^\perp}$ and $M_{\ker F_*}$ are integral manifolds of $(\ker F_*)^\perp$ and $\ker F_*$, respectively. Also f is a smooth function on M.*

Proof Suppose that $F : (M^m, g_M) \longrightarrow (N^n, g_N)$ be an umbilical conformal Riemannian map between Riemannian manifolds. Let $X, Y \in \Gamma((\ker F_*)^\perp)$ and $Z \in \Gamma(\ker F_*)$. We denote the second fundamental forms of $(\ker F_*)^\perp$ and $(\ker F_*)$ by $h^\perp$ and h respectively. Then we have

$$g_M(h^\perp(X, Y), Z) = g_M(\nabla^M_X Y, Z) = -g_M(Y, \nabla^M_X Z) = -g_M(Y, \mathcal{H}\nabla^M_X Z),$$

where $\mathcal{H}$ and ∇^M denote the projection onto $(\ker F_*)^\perp$ and Levi-Civita connection on M, respectively. Using (3.1), we get $g_M(h^\perp(X, Y), Z) = -\frac{1}{\lambda^2} g_N(F_* Y, F_*(\mathcal{H}\nabla^M_X Z))$. Then from (2.8), we derive

$$g_M(h^\perp(X, Y), Z) = -\frac{1}{\lambda^2} g_N\big(F_* Y, -(\nabla F_*)(X, Z) + \nabla^F_X F_*(Z)\big).$$

Since F is umbilical, we have $(\nabla F_*)(X, Z) = \frac{1}{m} g_M(X, Z)\tau = 0$. Thus it follows that $h^\perp(X, Y) = 0$ for every $X, Y \in \Gamma((\ker F_*)^\perp)$, that is, the distribution $(\ker F_*)^\perp$ is totally geodesic in M. On the other hand, for $X, Y \in \Gamma((\ker F_*)^\perp)$, we have $F_*([X, Y]) = [F_* X, F_* Y]$. Since $(\ker F_*)^\perp$ is totally geodesic in M, we get $[X, Y] \in \Gamma((\ker F_*)^\perp)$. So $F_*[X, Y] = [F_* X, F_* Y] \in \Gamma(\text{range } F_*)$. Thus range F_* is integrable.

We now show that the distribution $(\ker F_*)$ is an umbilical foliation in M. First note that $\ker F_*$ is integrable. By direct computation, we have $g_M(h(Z,V),X) = g_M(\nabla^M_Z V, X)$ for $Z, V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$. Then (3.1) implies that $g_M(h(Z,V),X) = \frac{1}{\lambda^2} g_N(F_*(\mathcal{H}\nabla^M_Z V), F_*(X))$. Using again (2.8) we have

$$g_M(h(Z,V),X) = \frac{1}{\lambda^2} g_N(-(\nabla F_*)(Z,V) + \nabla^F_Z F_*(V), F_*(X)).$$

Since $V \in \Gamma(\ker F_*)$ and F is umbilical, we obtain

$$g_M(h(Z,V),X) = -\frac{1}{m\lambda^2} g_M(Z,V) g_N(\tau, F_*(X)).$$

Using Lemma 4.2 in the above equation, we get

$$\begin{aligned} g_M(h(Z,V),X) = &-\frac{1}{m\lambda^2} g_M(Z,V)\{g_N(-(m-n_1)F_*(\mu^{\ker F_*}) \\ &+ (2-n_1)F_*(\mathrm{grad}(\ln\lambda)), F_*(X))\} \end{aligned}$$

due to $g_N(\mu^{\mathrm{range}\, F_*}, F_*(X)) = 0$. Using again (3.1) we obtain

$$\begin{aligned} g_M(h(Z,V),X) = &-\frac{1}{m} g_M(Z,V)\{-(m-n_1)g_M(\mu^{\ker F_*}, X) \\ &+ (2-n_1) g_M(\mathcal{H}(\mathrm{grad}(\ln\lambda)), X)\}. \end{aligned}$$

This shows that the distribution $(\ker F_*)$ is an umbilical distribution with the mean curvature vector field $-\frac{1}{m}\{-(m-n_1)\mu^{\ker F_*} + (2-n_1)\mathcal{H}(\mathrm{grad}(\ln\lambda))\}$. Hence proof is complete. □

Remark 5.1 We note that there is application of twisted product manifolds in general relativity. Indeed, the relationship between twisted products and spacetimes includes many well-known relativistic ones, such as Robertson-Walker, the stationary part of Reissner-Nordström, and the outer as well as the inner parts of Schwarzschild was obtained in [16].

We now recall that a map $F : (M_1^m, g_1) \longrightarrow M_2^n$ is called subimmersion at $p_1 \in M_1$ if there is a neighborhood U of p_1, a manifold P, a submersion $S : U \longrightarrow P$ and an immersion $I : P \longrightarrow M_2$ such that $F\,|_U = F_U = I \circ S$. A map $F : (M_1^m, g_1) \longrightarrow M_2^n$ is called subimmersion if it is subimmersion at each $p_1 \in M_1$. It is well known that $F : (M_1^m, g_1) \longrightarrow M_2^n$ is a subimmersion if and only if the rank of the linear map $F_{*p_1} : T_{p_1}M_1 \longrightarrow T_{F(p_1)}M_2$ is constant for p_1 in each connected component of M_1 [1], where M_1 and M_2 are finite dimensional manifolds. Thus by the definition, a proper conformal Riemannian map (as for Riemannian map) is a subimmersion. In a Riemannian map, (U, g_{1_U}) and (P, g_P) (where $g_P = I^* g_2$) are Riemannian manifolds, the submersion $S : (U, g_{1_U}) \longrightarrow (P, g_P)$ is a Riemannian submersion and the immersion $I : (P, g_P) \longrightarrow (M_2^n, g_2)$ is an isometric immersion (see: [13]). If $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is a proper conformal Riemannian map, then we have the following.

Theorem 5.2 *Let $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is a proper conformal Riemannian map between Riemannian manifolds (M_1^m, g_1) and (M_2^n, g_2). Let U, P, S and I be as in the definition of subimmersion so that $F\,|_U = F_U = I \circ S$. Let g_{1_U} denote the restriction of g_1 to U*

and let $g_P = I^* g_2$. *Then* (U, g_{1U}) *and* (P, g_P) *are Riemannian manifolds, the submersion* $S : (U, g_{1U}) \longrightarrow (P, g_P)$ *is a horizontally proper conformal submersion and the immersion* $I : (P, g_P) \longrightarrow (M_2^n, g_2)$ *is an isometric immersion.*

Proof Since U is open in M_1, (U, g_{1U}) is a Riemannian manifold. From Lemma 4.3.1 of [17], it follows that (P, g_P) is a Riemannian manifold and $I : (P, g_P) \longrightarrow (M_2^n, g_2)$ is an isometric immersion. Thus, it is enough to show that S is a horizontally conformal submersion. As in the [17], for each $p_1 \in U$, we define

$$S^h_{*p_1} : \mathcal{H}(p_1) \longrightarrow T_{S(p_1)} P \quad \text{by } S^h_{*p_1} X = S_{*p_1} X$$

and define

$$I^h_{*p_1} : T_{S(p_1)} P \longrightarrow \text{range } F_{*p_1} \quad \text{by } I^h_{*p_1} Z = I_{*p_1} Z.$$

Thus we have

$$F^h_{*p_1} = (I_{*S(p_1)} \circ S_{*p_1})^h = I^h_{*S(p_1)} \circ S^h_{*p_1} : \mathcal{H}(p_1) \longrightarrow \text{range } F_{*p_1}.$$

Then, for $X, Y \in \mathcal{H}(p_1)$ we have

$$g_2(F^h_{*p_1} X, F^h_{*p_1} Y) = g_2(I^h_{*S(p_1)} \circ S^h_{*p_1} X, I^h_{*S(p_1)} \circ S^h_{*p_1} Y).$$

Since $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is proper conformal Riemannian and $I : (P, g_P) \longrightarrow (M_2^n, g_2)$ is an isometric immersion, we get

$$\lambda^2 g_{1U}(X, Y) = g_P(S^h_{*p_1} X, S^h_{*p_1} Y)$$

which shows that $S : (U, g_{1U}) \longrightarrow (P, g_P)$ is a horizontally conformal submersion. □

Concluding Remarks A remarkable property of Riemannian maps is that a Riemannian map satisfies the generalized eikonal equation $\|F_*\|^2 = \text{rank}\, F$. Since the left hand side of this equation is continuous on the Riemannian manifold M and since rank F is an integer valued function, this equality implies that rank F is locally constant and globally constant on connected components. Thus if M is connected, the energy density $e(F) = \frac{1}{2}\|F_*\|^2$ is quantized to integer and half-integer values. The eikonal equation of geometrical optics solved by using Cauchy's method of characteristics, whereby, for real valued functions F, solutions to the partial differential equation $\|dF\|^2 = 1$ are obtained by solving the system of ordinary differential equations $x' = \text{grad}\, f(x)$. Since harmonic maps generalize geodesics, harmonic maps could be used to solve the generalized eikonal equation [13].

In [13], Fischer also proposed an approach to build a quantum model and he pointed out the success of such a program of building a quantum model of nature using Riemannian maps would provide an interesting relationship between Riemannian maps, harmonic maps and Lagrangian field theory on the mathematical side, and Maxwell's equation, Shrödinger's equation and their proposed generalization on the physical side.

As we have seen in the introduction and the above notes, there are many applications of isometric immersions, Riemannian submersions, Riemannian maps and conformal maps. As a generalization of all these maps, conformally Riemannian maps may have possible applications in mathematical physics, medical imaging theory and computer graphics.

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