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The Eikonal Equation of an Indefinite Metric

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Abstract. The concept of a semi-Riemannian map is introduced and it is shown that such maps are solutions of the eikonal equation. Also the existence of solutions to the eikonal equation are discussed and their relation to the Laplace–Beltrami equation is investigated.

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1. Introduction

The concept of a Riemannian map was introduced by Fischer [1] and it is shown that these maps are solutions of the (generalized) eikonal equation and, at the same time, they are Riemannian subimmersions. In this paper, we will generalize these maps to semi-Riemannian manifolds by introducing the concept of a semi-Riemannian map. Yet this definition of a semi-Riemannian map will be essentially different from a Riemannian map since tangent maps of such maps may have degenerate kernels as well as degenerate ranges, and they may not be subimmersions. First, we will define the nondegenerate rank of a map between semi-Riemannian manifolds and call it a solution of the (generalized) eikonal equation if it satisfies this equation with respect to its nondegenerate rank. Later, we will define a semi-Riemannian map as an isometry between its nondegenerate kernel and range. We will show that semi-Riemannian maps are solutions of the (generalized) eikonal equation, yet not every solution of this equation is a semi-Riemannian map. We will also obtain a necessary and sufficient condition on Lorentzian manifolds for the existence of solutions to the eikonal equation. At the same time, we will obtain some sufficient conditions on a solution of the eikonal equation to be a solution of the Laplace–Beltrami equation.

2. Preliminaries

Let E_1 and E_2 be inner product spaces (possibly indefinite) with inner products h_1 and h_2, respectively, and let $f\colon E_1 \to E_2$ be a linear map. The *transpose*

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${}^t f$: $E_2 \to E_1$ of f is defined by $h_1({}^t f x, y) = h_2(x, fy)$, where $x \in E_2$ and $y \in E_1$. The *indefinite square norm* $\|f\|^2$ of f with respect to metrics h_1, h_2 is defined by $\|f\|^2 = \mathrm{tr}({}^t f \circ f)$. Throughout this paper, let M always denote an n-dimensional connected semi-Riemannian manifold with metric g of index $0 \leqslant \nu \leqslant n$. A Lorentzian manifold M ($\nu = n-1$) is called *stably causal* if it is chronological with respect to a 'wider' metric (see [4, p. 1159]). Furthermore, stable causality is equivalent to the existence of a smooth function f: $M \to \mathbb{R}$ such that $g(\nabla f, \nabla f) > 0$ on M (cf. [4, p. 1159]). Note that the stable causality is a conformally invariant property of M, i.e., if M is stably causal, then M remains stably causal with respect to every metric conformal to g.

The rest of our notations and terminology will be the same as in [3].

3. The Eikonal Equation

Let $\mathcal{E}$ be either 0 or 1. A smooth function f: $M \to \mathbb{R}$ is said to satisfy the eikonal equation for $\mathcal{E}$ if $g(\nabla f, \nabla f) = \mathcal{E}$. (In local coordinates, $g^{ij}(\partial f/\partial x^i)(\partial f/\partial x^j) = \mathcal{E}$). Now we will determine the functions which are the solutions of the eikonal equation for $\mathcal{E}$.

DEFINITION 3.1. A smooth function f: $M \to \mathbb{R}$ is called semi-Riemannian at $p \in M$ if either $(\ker f_{*p})$ is a degenerate subspace of TpM or, otherwise, f_{*p}: $(\ker f_{*p})^\perp \to T_{f(p)}\mathbb{R}$ is an isometry or the zero map, where $(\ker f_{*p})^\perp$ has the induced metric and $\mathbb{R}$ has the usual positive definite metric. f is called semi-Riemannian if it is semi-Riemannian at each $p \in M$.

THEOREM 3.2. *f: $M \to \mathbb{R}$ is semi-Riemannian at $p \in M$ iff $g(\nabla f(p), \nabla f(p)) = \mathcal{E}$. Furthermore, if M is connected, then f is semi-Riemannian iff $g(\nabla f, \nabla f) = \mathcal{E}$ on M.*

Proof. First note that $g(\nabla f(p), \nabla f(p)) = 0$ iff either $(\ker f_{*p})$ is a degenerate subspace of TpM or otherwise f_{*p}: $(\ker f_{*p})^\perp \to T_{f(p)}\mathbb{R}$ is the zero map. Now suppose $g(\nabla f(p), \nabla f(p)) = 1$ (equivalently, $\ker f_{*p}$ is nondegenerate). Then

$$f_{*p}\colon \operatorname{span}\{\nabla f(p)\} \longrightarrow T_{f(p)}\mathbb{R}$$

satisfies

$$(f_{*p}(\nabla f(p))) \cdot (f_{*p}(\nabla f(p))) = \left(\mathrm{d}f_p(\nabla f(p))\right)^2 = g(\nabla f(p), \nabla f(p))^2.$$

Hence, f_{*p} is an isometry iff

$$g(\nabla f(p), \nabla f(p)) = g(\nabla f(p), \nabla f(p))^2.$$

Thus, it follows that $g(\nabla f(p), \nabla f(p)) = \mathcal{E}$ iff f is semi-Riemannian at p, where $\mathcal{E} = 0, 1$. Also, if M is connected, then f is semi-Riemannian on M iff $g(\nabla f(p), \nabla f(p)) = \mathcal{E}$ on M since $g(\nabla f, \nabla f)$ is a continuous function on M. □

Remark. Observe here that $g(\nabla f, \nabla f) = \|f_*\|^2$. That is, f is semi-Riemannian iff $\|f_*\|^2 = \mathcal{E}$.

It is natural to ask the necessary and sufficient conditions for the existence of solutions to the eikonal equation. We will only give a partial answer to this question for Lorentzian manifolds, where the solutions of the Eikonal equation are related to the causal structure of Lorentzian manifolds.

THEOREM 3.3. *The eikonal equation $\mathcal{E} = 1$ has a solution with respect to a conformal metric g_c on M iff the inequality $g(\nabla f, \nabla f) > 0$ has a solution.*

Proof. Note that if $g_c = \varphi g$ is a conformal metric on M, then the gradient $\overset{c}{\nabla}$ with respect to g_c is $\overset{c}{\nabla} = (1/\varphi)\nabla$. Hence, if $g_c(\overset{c}{\nabla} f, \overset{c}{\nabla} f) = 1$, then $g(\nabla f, \nabla f) = \varphi > 0$. Conversely, consider the conformal metric $g_c = g(\nabla f, \nabla f) g$ on M. Then it is easy to show that f is a solution of $g_c(\overset{c}{\nabla} f, \overset{c}{\nabla} f) = 1$, where $\overset{c}{\nabla}$ is the gradient with respect to g_c. Note that

$$\overset{c}{\nabla} f = \frac{1}{g(\nabla f, \nabla f)} \nabla f. \qquad \square$$

COROLLARY 3.4. *Let M be a Lorentzian manifold. M is stably causal iff the eikonal equation for $\mathcal{E} = 1$ has a solution with respect to a conformal metric.*

Proof. Immediate by Theorem 3.3, and the definition of stable causality. □

Next, we will try to find a relationship between the eikonal and Laplace–Beltrami equations on two-dimensional Lorentzian manifolds.

LEMMA 3.5. *A map $f\colon M \to \mathbb{R}$ is semi-Riemannian (i.e. a solution of the eikonal equation for $\mathcal{E}$) iff ∇f is a geodesic vector field and is unit provided that ∇f is nonnull (i.e. $\mathcal{E} = 1$).*

Proof. Let $X \in \Gamma TM$. Then since

$$\begin{aligned} Xg(\nabla f, \nabla f) &= 2g(\nabla_X \nabla f, \nabla f) \\ &= 2H_f(X, \nabla f) \\ &= 2H_f(\nabla f, X) \\ &= 2g(\nabla_{\nabla f} \nabla f, X), \end{aligned}$$

it follows that $\nabla_{\nabla f} \nabla f = 0$ iff $g(\nabla f, \nabla f)$ is constant, where H_f is the Hessian of f. Hence, ∇f is a geodesic vector field and is unit provided that ∇f is nonnull, iff f satisfies the Eikonal equation. □

COROLLARY 3.6. *Let M be a two-dimensional Lorentzian manifold. If $f\colon M \to \mathbb{R}$ is a solution of the eikonal equation for $\mathcal{E} = 0$ with $\nabla f \neq 0$ at each $p \in M$, then f is also a solution of the Laplace–Beltrami equation on M.*

Proof. First note that $U_1 = \nabla f$ is a geodesic null vector field on M. Also, let U_2 be a null vector field on M with $g(U_1, U_2) = -1/2$. Then $Z_1 = U_1 + U_2$ and $Z_2 = U_1 - U_2$ is an orthonormal basis for TM. Hence, by Lemma 3.5,

$$\begin{aligned}\Delta f = \operatorname{div} \nabla f &= -g(\nabla_{Z_1} U_1, Z_1) + g(\nabla_{Z_2} U_1, Z_2) \\ &= -g(\nabla_{U_2} U_1, U_2) + g(\nabla_{U_2} U_1, U_2) \\ &= 0.\end{aligned}$$

□

The converse of the above corollary is not generally true. For example, consider $M = S^1 \times \mathbb{R}$ with the product metric $g = -\mathrm{d}\theta \otimes \mathrm{d}\theta \oplus \mathrm{d}t \otimes \mathrm{d}t$, where θ is the polar coordinate on S^1 and t is the usual coordinate on $\mathbb{R}$. This Lorentzian manifold is not stably causal (since it contains closed timelike curves) and, hence, $g(\nabla f, \nabla f) = 0$ does not have a solution with $\nabla f \neq 0$ at each $p \in M$ (cf. [5, p. 258]), but the Laplace–Beltrami equation has solutions on M, i.e., the general solution is of the form $\varphi_1(\theta + t) + \varphi_2(\theta - t)$, where $\varphi_1, \varphi_2 \colon \mathbb{R} \to \mathbb{R}$ smooth periodic functions with period 2π.

COROLLARY 3.7. *If* $f \colon M \to \mathbb{R}$ *is a solution of* $g(\nabla f, \nabla f) = 1$, *then* f *is a semi-Riemannian submersion* (see [3, p. 212]). *Furthermore, the fibers* $N(c) = f^{-1}(c)$ *are totally geodesic iff* $H_f = 0$ *on* $N(c)$.

Proof. The first claim is immediate from Theorem 3.2. For the second claim, let $X, Y \in \Gamma TN(c)$. Then, since

$$\begin{aligned}g(\nabla_X Y, \nabla f) &= Xg(Y, \nabla f) - g(Y, \nabla_X \nabla f) \\ &= -H_f(X, Y),\end{aligned}$$

$\nabla_X Y \in \Gamma TN(c)$ iff $H_f = 0$ on $N(c)$. □

COROLLARY 3.8. *If* $f \colon M \to \mathbb{R}$ *is a solution of* $g(\nabla f, \nabla f) = 0$ *with* $\nabla f \neq 0$ *at each* $p \in M$, *then* $N(c) = f^{-1}(c)$ *is a degenerate hypersurface in* M. *Furthermore,* $N(c)$ *is totally geodesic iff* $H_f = 0$ *on* $N(c)$.

Proof. Similar to the proof of Corollary 3.7. □

Remark. Note that, if f is a solution of the eikonal equation for $\mathcal{E}$, then since ∇f is a geodesic vector field, $H_f = 0$ on each fiber $N(c)$ is equivalent to $H_f = 0$ on M.

COROLLARY 3.9. *If* $f \colon M \to \mathbb{R}$ *is a solution of the eikonal equation for* $\mathcal{E}$ *with totally geodesic fibers, then* f *is also a solution of the Laplace–Beltrami equation.*

Proof. Immediate from the above remark, since

$$\Delta f = \operatorname{div} \nabla f = \operatorname{tr} H_f = 0.$$

□

It is natural to ask whether the eikonal equation is a kind of wave equation in General Relativity. Indeed, it may since the gravitational plane waves in Gravitational Plane Wave solutions in General Relativity satisfy the eikonal equation for $\mathcal{E} = 0$ (see [5, p. 244]). Furthermore, gravitational plane waves satisfy the assumptions of the above corollary and hence, they are also solutions of the Laplace–Beltrami equation which is classically considered to be the wave equation (also see [2, p. 971]).

4. The Generalized Eikonal Equation

Let M_1 and M_2 be semi-Riemannian manifolds with metrics g_1 and g_2, respectively, and let $f\colon M_1 \to M_2$ be a smooth function. Also, at each $p \in M_1$ and $q = f(p) \in M_2$, define

$$L_1(p) = (\ker f_{*p}) \cap (\ker f_{*p})^{\perp} \subseteq TpM_1$$

and

$$L_2(q) = (\text{range}\, f_{*p}) \cap (\text{range}\, f_{*p})^{\perp} \subseteq TqM_2.$$

Note here that $L_1(p)$ and $L_2(q)$ are the degenerate spaces of the restrictions of the metrics q_{1_p} and g_{2_q} to $(\ker f_{*p})^{\perp}$ and $(\text{range}\, f_{*p})$, respectively. Also define the quotient spaces $A(p)$ and $B(q)$ by

$$A(p) = (\ker f_{*p})^{\perp}/L_1(p) \quad \text{and} \quad B(q) = (\text{range}\, f_{*p})/L_2(q).$$

Now define nondegenerate inner products $\bar{g}_{1_p}$ and $\bar{g}_{2_q}$ in $A(p)$ and $B(q)$, respectively, as follows:

(1) $\bar{g}_{1_p}(\bar{x}, \bar{y}) = g_{1_p}(x, y)$, where $x, y \in (\ker f_{*p})^{\perp}$ with $\pi_1(x) = \bar{x}$, $\pi_1(y) = \bar{y}$ and $\pi_1\colon (\ker f_{*p})^{\perp} \to A(p)$ is the canonical projection.

(2) $\bar{g}_{2_q}(\bar{x}, \bar{y}) = g_{2_q}(x, y)$, where $x, y \in \text{range}\, f_{*p}$ with $\pi_2(x) = \bar{x}$, $\pi_2(y) = \bar{y}$ and $\pi_2\colon (\text{range}\, f_{*p}) \to B(q)$ is the canonical projection.

Finally define the linear map $\bar{f}_{*p}\colon A(p) \to B(q)$ by $\bar{f}_{*p}(\bar{x}) = \pi_2(f_{*p}x)$, where $x \in (\ker f_{*p})^{\perp}$ with $\pi_1(x) = \bar{x}$. (It is easy to check that $\bar{g}_{1_p}, \bar{g}_{2_q}, \bar{f}_{*p}$ are well defined.) Here note that $\bar{f}_{*p}$ may not be an isomorphism, it may not even be injective or surjective unlike the Riemannian case (see [1]).

DEFINITION 4.1. Let $f\colon M_1 \to M_2$ be a smooth map. The nondegenerate rank of f at $p \in M_1$ is defined to be the rank $\bar{f}_{*p}$.

Remark. Note that $\text{rank}\, \bar{f}_{*p} = \text{rank}\, f_{*p} - \dim L_1(p)$. Here, observe that $A = \bigcup_{p \in M_1} A(p)$ need not be a vector bundle in general. In fact, even if $\text{rank}\, \bar{f}_*$ is constant, A still may not be a continuous vector bundle over M_1.

LEMMA 4.2. *If $f\colon M_1 \to M_2$ is a smooth map, then $\|f_{*p}\|^2 = \|\bar{f}_{*p}\|^2$, where the first indefinite square norm is with respect to the metrics g_{1_p}, g_{2_q} and the second is with respect to $\bar{g}_{1_p}$, $\bar{g}_{2_q}$.*

Proof. Let $\{x_1, \dots, x_\ell\}$ be an orthonormal basis for a nondegenerate complementary space to $L_1(p)$ in $(\ker f_{*p})$ and let $\{y_1, \dots, y_m\}$ be an orthonormal basis for a nondegenerate complementary space to $L_1(p)$ in $(\ker f_{*p})^\perp$. Also, let $\{z_1, w_1, \dots, z_k, w_k\}$ be an orthonormal basis for $(\operatorname{span}\{x_1, \dots, x_\ell, y_1, \dots, y_m\})^\perp$ such that $u_i = z_i + w_i \in L_1(p)$ for $i = 1, \dots, k$ (cf. [2, p. 53]). Then $\{z_1, w_1, \dots, z_k, w_k, x_1, \dots, x_\ell, y_1, \dots, y_m\}$ is an orthonormal basis for TpM_1 and, hence,

$$\begin{aligned}\|f_{*p}\|^2 &= \sum_{i=1}^{k} g_{1_p}(z_i, z_i) g_{1_p}({}^t f_{*p} \circ f_{*p} z_i, z_i) + \\ &\quad + \sum_{i=1}^{k} g_{1_p}(w_i, w_i) q_{1_p}({}^t f_{*p} \circ f_{*p} w_i, w_i) + \\ &\quad + \sum_{i=1}^{\ell} g_{1_p}(x_i, x_i) g_{1_p}({}^t f_{*p} \circ f_{*p} x_i, x_i) + \\ &\quad + \sum_{i=1}^{m} g_{1_p}(y_i, y_i) g_{1_p}({}^t f_{*p} \circ f_{*p} y_i, y_i) \\ &= \sum_{i=1}^{k} g_{1_p}(z_i, z_i) g_{2_q}(f_{*p} z_i, f_{*p} z_i) + \\ &\quad + \sum_{i=1}^{k} g_{1_p}(w_i, w_i) g_{2_q}(f_{*p} w_i, f_{*p} w_i) + \\ &\quad + \sum_{i=1}^{m} g_{1_p}(y_i, y_i) g_{2_q}(f_{*p} y_i, f_{*p} y_i).\end{aligned}$$

But since $u_i = z_i + w_i \in L_1(p) \subseteq \ker f_{*p}$,

$$0 = f_{*p}(u_i) = f_{*p}(z_i) + f_{*p}(w_i)$$

and

$$0 = g_{1_p}(u_i, u_i) = g_{1_p}(z_i, z_i) + g_{1_p}(w_i, w_i).$$

Hence

$$f_{*p}(z_i) = -f_{*p}(w_i) \quad \text{and} \quad g_{1_p}(z_i, z_i) = -g_{1_p}(w_i, w_i)$$

and it follows that

$$\|f_{*p}\|^2 = \sum_{i=1}^{m} g_{1_p}(y_i, y_i) g_{2_q}(f_{*p} y_i, f_{*p} y_i)$$

$$= \sum_{i=1}^{m} \bar{g}_{1_p}(\bar{y}_i, \bar{y}_i)\bar{g}_{2_q}(\bar{f}_{*p}\bar{y}_1, \bar{f}_{*p}\bar{y}_i)$$
$$= \|\bar{f}_{*p}\|^2. \qquad \square$$

A function $f\colon M_1 \to M_2$ is said to satisfy the *generalized eikonal equation* if $\|f_*\|^2 = \operatorname{rank} \bar{f}_*$. Here, note that since $\|f_*\|^2$ is a continuous function on M_1, if M_1 is connected, then $\operatorname{rank} \bar{f}_*$ is necessarily constant on M_1. But note that the constancy of $\operatorname{rank} \bar{f}_*$ does not imply the constancy of $\operatorname{rank} f_*$.

DEFINITION 4.3. A smooth map $f\colon M_1 \to M_2$ is called semi-Riemannian at p if $\bar{f}_{*p}\colon A(p) \to B(q)$ is an (into) isometry. f is called semi-Riemannian if it is semi-Riemannian at each $p \in M$.

Remark. Note that the definition of a semi-Riemannian map in Definition 3.1, could also be stated as in Definition 3.2. Also, the eikonal equation of Section 3 could also be written in the above form.

Now we will show that semi-Riemannian maps are solutions of the generalized eikonal equation.

THEOREM 4.4. *If $f\colon M_1 \to M_2$ is semi-Riemannian at $p \in M$, then $\|f_{*p}\|^2 = \operatorname{rank} \bar{f}_{*p}$. Furthermore, if M_1 is connected and f is semi-Riemannian on M_1, then f satisfies the generalized eikonal equation.*

Proof. First, we will show that $\bar{F}_p = {}^t\bar{f}_{*p} \circ \bar{f}_{*p}\colon A(p) \to A(p)$ is the identity map. For, note that for every $\bar{x}, \bar{y} \in A(p)$,

$$\begin{aligned} \bar{g}_{1_p}(\bar{F}_p\bar{x}, \bar{y}) &= \bar{g}_{2_q}(\bar{f}_{*p}\bar{x}, \bar{f}_{*p}\bar{y}) \\ &= \bar{g}_{1_p}(\bar{x}, \bar{y}). \end{aligned}$$

Hence, $\bar{F}_p = \overline{\mathrm{id}}$ and it follows from Lemma 4.2 that

$$\|f_{*p}\|^2 = \|\bar{f}_{*p}\|^2 = \operatorname{tr} \bar{F}_p = \dim A(p) = \operatorname{rank} \bar{f}_*. \qquad \square$$

Remark. Note that semi-Riemannian maps, unlike to Riemannian maps in the definite case, need not be semi-Riemannian subimmersions (see [1]). Also, the generalized eikonal equation as well may have solutions which are not semi-Riemannian maps unlike the eikonal equation of Section 3. For example, consider Minkowski space $\mathbb{R}^3 = M_1$ and Euclidean space $\mathbb{R}^2 = M_2$. (Here M_1 has the metric $g_1 = -\mathrm{d}x \otimes \mathrm{d}x - \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z$ and M_2 has the metric $g_2 = \mathrm{d}u \otimes \mathrm{d}u + \mathrm{d}v \otimes \mathrm{d}v$.) Then it can be easily shown that the map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f(x, y, z) = (x + \sqrt{2}z, y + \sqrt{2}z)$ has rank 2 and nondegenerate $(\ker f_*)$. Thus, $2 = \operatorname{rank} f = \operatorname{rank} \bar{f}_*$. Also, with an easy computation, it can be shown that f satisfies the generalized eikonal equation $\|f_*\|^2 = \operatorname{rank} \bar{f}_*$ $(= 2)$. Yet note that f is not a semi-Riemannian map, since $(\ker f_*)^{\perp}$ cannot be isometric to range f_*.

(The induced metric on $(\ker f_*)^\perp$ is indefinite yet the induced metric on range f_* is definite.) In fact observe that there exists no semi-Riemannian map satisfying the generalized eikonal equation for this case ($\operatorname{rank} \bar{f}_* = 2$). Also note that the map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x,y,z) = (x + \sqrt{2}z, 0)$ is not semi-Riemannian, yet satisfies $\|f_*\|^2 = \operatorname{rank} \bar{f}_* = 1$. However, there are semi-Riemannian maps $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ satisfying $\|f_*\|^2 = \operatorname{rank} \bar{f}_* = 1$, for example, $f(x,y,z) = (0,z)$.

We can also state and prove a theorem out of the above Remark.

THEOREM 4.5. *Let $f = (f_1, \dots, f_k)\colon M \to \mathbb{R}^k$ be a smooth map, where $\mathbb{R}^k$ has the usaul Euclidean metric h. If* $\ker f_*$ *is nondegenerate, $\{\nabla f_1, \dots, \nabla f_k\}$ are linearly independent at each $p \in M$ and satisfy the eikonal equation $g(\nabla f_i, \nabla f_i) = 1$, then f satisfies the generalized eikonal equation $\|f_*\|^2 = \operatorname{rank} \bar{f}_* = k$.*

Proof. Clearly $(\ker f_*)$ is nondegenerate and $\operatorname{rank} f = \operatorname{rank} \bar{f}_* = k$. To show that f satisfies the generalized eikonal equation, let $\{x_1, \dots, x_n\}$ be a local orthonormal basis for TM. Then

$$\begin{aligned}
\|f_*\|^2 &= \sum_{i=1}^{n} g(x_i, x_i) h(f_* x_i, f_* x_i) \\
&= \sum_{i=1}^{n} g(x_i, x_i) h\big((\mathrm{d}f_1(x_i), \dots, \mathrm{d}f_k(x_i)), (\mathrm{d}f_1(x_i), \dots, \mathrm{d}f_k(x_i))\big) \\
&= \sum_{i=1}^{n} g(x_i, x_i)\big((\mathrm{d}f_1(x_i))^2 + \cdots + (\mathrm{d}f_k(x_i))^2\big) \\
&= g(\nabla f_1, \nabla f_1) + \cdots + g(\nabla f_k, \nabla f_k) \\
&= k = \operatorname{rank} \bar{f}_*.
\end{aligned}$$

□

Finally, we will give an example of a semi-Riemannian map with degenerate kernel. Let $\mathbb{R}^4$ has the semi-Euclidean metric

$$g = -\mathrm{d}x \otimes \mathrm{d}x - \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z + \mathrm{d}t \otimes \mathrm{d}t$$

and $\mathbb{R}^3$ has the semi-Euclidean metric

$$h = -\mathrm{d}u \otimes \mathrm{d}u + \mathrm{d}v \otimes \mathrm{d}v + \mathrm{d}w \otimes \mathrm{d}w.$$

Then it can be easily that the map $f\colon \mathbb{R}^4 \to \mathbb{R}^3$, given by $f(x,y,z,t) = (x+z, x+z, t)$, is a semi-Riemannian map with $\dim L_1 = 1$, $\operatorname{rank} f = 2$ and $\operatorname{rank} \bar{f}_* = 1$. Since this map is semi-Riemannian, it satisfies $\|f_*\|^2 = \operatorname{rank} \bar{f}_* = 1$.

Remark. From Theorem 3.2, we know that a map $f\colon M \to \mathbb{R}$ is a solution of the eikonal equation iff it is semi-Riemannian. Yet, as we see from the examples above, the same is not true for the generalized eikonal equation. Although semi-Riemannian maps are solutions of the generalized eikonal equation, a solution of this equation may not be semi-Riemannian. Hence, among all solutions of the generalized eikonal equation, semi-Riemannian maps constitute a special subset which has the geometric significance of being partial isometries.

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