Isometry of Kaehlerian manifolds to complex projective spaces

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§1. Introduction.

Let M be a complex n-dimensional connected Kaehlerian manifold covered by a system of real coordinate neighborhoods $\{U; x^h\}$, where, here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$ and let g_{ji}, $F_i{}^h$, $\{_j{}^h{}_i\}$, ∇_i, $K_{kji}{}^h$, K_{ji} and K be respectively the Hermitian metric tensor, the complex structure tensor, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to $\{_j{}^h{}_i\}$, the curvature tensor, the Ricci tensor and the scalar curvature of M.

A vector field v^h is called a *holomorphically projective* (or *H-projective*, for brevity) vector field [2, 3, 5, 7] if it satisfies

$$(1.1) \qquad L_v\{_j{}^h{}_i\} = \nabla_j \nabla_i v^h + v^k K_{kji}{}^h$$

$$= \delta_j^h \rho_i + \delta_i^h \rho_j - \rho_t F_j{}^t F_i{}^h - \rho_t F_i{}^t F_j{}^h$$

for a certain covariant vector field ρ_i on M, called the associated covariant vector field of v^h, where L_v denotes the operator of Lie derivation with respect to v^h. In particular, if ρ_i is zero vector field then v^h is called an *affine* vector field. When we refer in the sequel to an *H*-projective vector field v^h, we always mean by ρ_i the associated covariant vector field appearing in (1.1).

Recently, the present authors [9, 10] and one of the present authors [1] proved a series of integral inequalities in a compact Kaehlerian manifold with constant scalar curvature admitting an *H*-projective vector field and then obtained necessary and sufficient conditions for such a Kaehlerian manifold to be isometric to a complex projective space with Fubini-Study metric.

The purpose of the present paper is to continue the joint work [9, 10] of the present authors and to prove the following theorem.

THEOREM A. *If a complex $n > 1$ dimensional, compact, connected and simply connected Kaehlerian manifold M with constant scalar curvature K admits a non-affine H-projective vector field v^h, then M is isometric to a complex projective space CP^n with Fubini-Study metric and of constant holomorphic sectional cur-*

vature $\dfrac{K}{n(n+1)}$.

In the sequel, we need the following theorem due to Obata [4]. (See also [6].)

THEOREM B. *Let M be a complete, connected and simply connected Kaehlerian manifold. In order for M to admit a non-trivial solution φ of a system of partial differential equations*

$$(1.2) \qquad \nabla_j \nabla_i \varphi_h + \frac{c}{4}(2\varphi_j g_{ih} + \varphi_i g_{jh} + \varphi_h g_{ji} - F_{ji}F_h{}^t\varphi_t - F_{jh}F_i{}^t\varphi_t) = 0 ,$$

where $\varphi_h = \nabla_h \varphi$ and $F_{ji} = F_j{}^t g_{ti}$, c being a positive constant, it is necessary and sufficient that M is isometric to a complex projective space CP^n with Fubini-Study metric and of constant holomorphic sectional curvature c.

We assume in this paper that Kaehlerian manifolds under consideration are connected.

§ 2. Preliminaries.

Let M be a complex n-dimensional Kaehlerian manifold. The complex structure tensor $F_i{}^h$ and the Hermitian metric tensor g_{ji} of M satisfy

$$(2.1) \qquad F_t{}^h F_i{}^t = -\delta_i^h , \quad \nabla_j F_i{}^h = 0 , \quad \nabla_j F_{ih} = 0 ,$$

$$(2.2) \qquad g_{jt}F_i{}^t + g_{ti}F_j{}^t = 0$$

and

$$(2.3) \qquad g_{ji} - g_{ts}F_j{}^t F_i{}^s = 0 .$$

We have [7, 9, 10], for the curvature tensor $K_{kji}{}^h$,

$$(2.4) \qquad K_{kji}{}^t F_t{}^h - K_{kjt}{}^h F_i{}^t = 0 ,$$

$$(2.5) \qquad K_{kji}{}^h + K_{kjt}{}^s F_i{}^t F_s{}^h = 0 ,$$

$$(2.6) \qquad K_{kjit}F_h{}^t + K_{kjth}F_i{}^t = 0$$

and

$$(2.7) \qquad K_{kjih} - K_{kjts}F_i{}^t F_h{}^s = 0 ,$$

where $K_{kjih} = K_{kji}{}^t g_{th}$.

Using (2.6) and the first Bianchi identity

$$K_{kjih} + K_{ikjh} + K_{jikh} = 0 ,$$

we have

$$2K_{kt}F_j{}^t = 2g^{ut}K_{kuts}F_j{}^s = -2g^{ut}K_{kusj}F_t{}^s$$

$$= -2K_{ktsj}F^{ts} = -(K_{ktsj} - K_{kstj})F^{ts}$$

$$= K_{tskj}F^{ts} = K_{kjts}F^{ts},$$

from which

(2.8) $$K_{kjts}F^{ts} = 2K_{kt}F_j{}^t,$$

(2.9) $$K_{tjis}F^{ts} = -K_{jt}F_i{}^t$$

and

(2.10) $$K_{ktsh}F^{ts} = -K_{kt}F_h{}^t,$$

g^{ji} being contravariant components of g_{ji} and $F^{kj} = g^{kt}F_t{}^j$.

From (2.8), we have, for the Ricci tensor $K_{ji} = K_{tji}{}^t$,

(2.11) $$K_{jt}F_i{}^t + K_{ti}F_j{}^t = 0,$$

(2.12) $$K_{ji} - K_{ts}F_j{}^t F_i{}^s = 0,$$

(2.13) $$K_t{}^h F_i{}^t - K_i{}^t F_t{}^h = 0$$

and

(2.14) $$K_i{}^h + K_t{}^s F_i{}^t F_s{}^h = 0,$$

where $K_i{}^h = K_{it}g^{th}$.

A Kaehlerian manifold M has the constant holomorphic sectional curvature k if and only if

(2.15) $$K_{kji}{}^h = \frac{k}{4}(\delta_k^h g_{ji} - \delta_j^h g_{ki} + F_k{}^h F_{ji} - F_j{}^h F_{ki} - 2F_{kj}F_i{}^h).$$

We define tensor fields G_{ji} and $Z_{kji}{}^h$ [9, 10] on M by

(2.16) $$G_{ji} = K_{ji} - \frac{K}{2n}g_{ji}$$

and

(2.17) $$Z_{kji}{}^h = K_{kji}{}^h - \frac{K}{4n(n+1)}(\delta_k^h g_{ji} - \delta_j^h g_{ki} + F_k{}^h F_{ji} - F_j{}^h F_{ki} - 2F_{kj}F_i{}^h)$$

respectively. If $G_{ji} = 0$ for $n > 1$ then M is a Kaehler-Einstein manifold and K is a constant and if $Z_{kji}{}^h = 0$ for $n > 1$ then M is of constant holomorphic sectional curvature $\dfrac{K}{n(n+1)}$.

We easily see that the tensor fields G_{ji} and $Z_{kji}{}^h$ satisfy

(2.18) $$G_{ji} = G_{ij}, \quad G_{ji}g^{ji} = 0, \quad Z_{tji}{}^t = G_{ji},$$

(2.19) $Z_{kjih} = -Z_{jkih}, \quad Z_{kjih} = Z_{ihkj}$

and

(2.20) $Z_{kji}{}^h + Z_{ikj}{}^h + Z_{jik}{}^h = 0,$

where $Z_{kjih} = Z_{kji}{}^t g_{th}$.

The tensor fields G_{ji} and $Z_{kji}{}^h$ also satisfy

(2.21) $G_{jt}F_i{}^t + G_{ti}F_j{}^t = 0,$

(2.22) $G_{ji} - G_{ts}F_j{}^t F_i{}^s = 0,$

(2.23) $G_t{}^h F_i{}^t - G_i{}^t F_t{}^h = 0,$

(2.24) $G_i{}^h + G_t{}^s F_i{}^t F_s{}^h = 0,$

(2.25) $Z_{kji}{}^t F_t{}^h - Z_{kjt}{}^h F_i{}^t = 0,$

(2.26) $Z_{kji}{}^h + Z_{kjt}{}^s F_i{}^t F_s{}^h = 0,$

(2.27) $Z_{kjit}F_h{}^t + Z_{kjth}F_i{}^t = 0,$

(2.28) $Z_{kjih} - Z_{kjts}F_i{}^t F_h{}^s = 0,$

(2.29) $Z_{kjts}F^{ts} = 2G_{kt}F_j{}^t,$

(2.30) $Z_{tjis}F^{ts} = -G_{jt}F_i{}^t$

and

(2.31) $Z_{ktsh}F^{ts} = -G_{kt}F_h{}^t,$

where $G_i{}^h = G_{it}g^{th}$.

If the scalar curvature K is a constant, then, from (2.1), (2.17) and the second Bianchi identity

$$\nabla_l K_{kji}{}^h + \nabla_j K_{lki}{}^h + \nabla_k K_{jli}{}^h = 0,$$

we have

(2.32) $\nabla_l Z_{kji}{}^h + \nabla_j Z_{lki}{}^h + \nabla_k Z_{jli}{}^h = 0,$

from which and (2.18) and (2.19),

(2.33) $\nabla_t Z_{kji}{}^t = \nabla_k G_{ji} - \nabla_j G_{ki}.$

A vector field u^h on M is said to be *contravariant analytic* if

(2.34) $(\nabla_j u_t)F_i{}^t + (\nabla_t u_i)F_j{}^t = 0$

or equivalently if

(2.35) $\nabla_j u_i - (\nabla_t u_s)F_j{}^t F_i{}^s = 0,$

where $u_i = g_{ih}u^h$. A vector field u^h on M is contravariant analytic if and only if

$$(2.36) \qquad L_u F_i{}^h = 0,$$

where L_u denotes the operator of Lie derivation with respect to u^h. It is known [7] that if M is compact then a necessary and sufficient condition for a vector field u^h on M to be contravariant analytic is that

$$(2.37) \qquad \nabla^i \nabla_i u^h + K_i{}^h u^i = 0$$

holds, where $\nabla^i = g^{ih}\nabla_h$.

For an H-projective vector field v^h on M defined by (1.1), we have

$$(2.38) \qquad \nabla_j \nabla_t v^t = 2(n+1)\rho_j$$

and

$$(2.39) \qquad \nabla^i \nabla_i v^h + K_i{}^h v^i = 0.$$

(2.38) shows that the associated covariant vector field ρ_j is gradient. Putting

$$(2.40) \qquad \rho = \frac{1}{2(n+1)}\nabla_t v^t,$$

we have

$$(2.41) \qquad \rho_j = \nabla_j \rho.$$

We have, for the H-projective vector field v^h, from (1.1),

$$(2.42) \qquad \nabla_j L_v g_{ih} = 2\rho_j g_{ih} + \rho_i g_{jh} + \rho_h g_{ji} - F_{ji}F_h{}^t \rho_t - F_{jh}F_i{}^t \rho_t,$$

from which

$$(2.43) \qquad \nabla^j L_v g_{ih} = 2\rho^j g_{ih} + \rho_i \delta_h^j + \rho_h \delta_i^j + F_i{}^j F_h{}^t \rho_t + F_h{}^j F_i{}^t \rho_t,$$

$$(2.44) \qquad \nabla_j L_v g^{ih} = -2\rho_j g^{ih} - \rho^i \delta_j^h - \rho^h \delta_j^i + F_j{}^i F^{ht}\rho_t + F_j{}^h F^{it}\rho_t$$

and

$$(2.45) \qquad \nabla^j L_v g^{ih} = -2\rho^j g^{ih} - \rho^i g^{jh} - \rho^h g^{ji} + F^{ji}F^{ht}\rho_t + F^{jh}F^{it}\rho_t,$$

where $\rho^h = \rho_i g^{ih}$.

Substituting (1.1) into the well known formula [7, 8]

$$L_v K_{kji}{}^h = \nabla_k L_v \{_j{}^h{}_i\} - \nabla_j L_v \{_k{}^h{}_i\},$$

we find

$$(2.46) \qquad \begin{aligned} L_v K_{kji}{}^h = {}&-\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i \\ &+ (F_k{}^h \nabla_j \rho_t - F_j{}^h \nabla_k \rho_t)F_i{}^t + (F_k{}^t \nabla_j \rho_t - F_j{}^t \nabla_k \rho_t)F_i{}^h, \end{aligned}$$

from which, contracting with respect to h and k,

(2.47) $L_v K_{ji} = -2n\nabla_j \rho_i - 2(\nabla_t \rho_s) F_j{}^t F_i{}^s$.

Suppose that an H-projective vector field v^h on M is contravariant analytic. Then, applying the operator L_v of Lie derivation with respect to v^h to both sides of (2.12), we have

$$L_v K_{ji} = (L_v K_{ts}) F_j{}^t F_i{}^s ,$$

from which and (2.47), we see that if $n > 1$ then ρ^h is also contravariant analytic and

(2.48) $L_v K_{ji} = -2(n+1)\nabla_j \rho_i$

holds.

If a Kaehlerian manifold M is compact, then we see, from (2.39), that an H-projective vector field v^h on M is contravariant analytic or equivalently $L_v F_i{}^h = 0$ holds and moreover if $n > 1$ then the associated vector field ρ^h is also contravariant analytic and (2.48) holds.

For a contravariant analytic H-projective vector field v^h on a complex $n > 1$ dimensional Kaehlerian manifold M with constant scalar curvature K, we have [9, 10], for the tensor field G_{ji},

(2.49) $L_v G_{ji} = -\nabla_j w_i - \nabla_i w_j$,

where we have put

(2.50) $$w^h = (n+1)\rho^h + \frac{K}{2n} v^h$$

and $w_i = g_{ih} w^h$, and, for the tensor field $Z_{kji}{}^h$,

(2.51) $L_v Z_{kji}{}^h = \dfrac{1}{2(n+1)} \{ \delta_k^h L_v G_{ji} - \delta_j^h L_v G_{ki} - F_k{}^h (L_v G_{jt}) F_i{}^t + F_j{}^h (L_v G_{kt}) F_i{}^t$

$\qquad\qquad\qquad\qquad - F_k{}^t (L_v G_{jt}) F_i{}^h + F_j{}^t (L_v G_{kt}) F_i{}^h \}$.

§ 3. Proof of Theorem A.

In this section, we prove Theorem A. For this purpose, we need a series of lemmas. We use freely formulas (2.1)∼(2.51) in the proofs of all lemmas and Theorem A in this section.

LEMMA 1 (Yano and Hiramatu [9]). *If, in a compact Kaehlerian manifold M, a non-constant function φ satisfies*

(3.1) $\nabla_j \nabla_i \varphi_h + \dfrac{c}{4}(2\varphi_j g_{ih} + \varphi_i g_{jh} + \varphi_h g_{ji} - F_{ji} F_h{}^t \varphi_t - F_{jh} F_i{}^t \varphi_t) = 0$,

where $\varphi_h = \nabla_h \varphi$, c being a constant, then the constant c is necessarily positive.

PROOF. Transvecting (3.1) with g^{ih}, we have

$$\nabla_j \Delta\varphi + (n+1)c\varphi_j = 0,$$

from which

$$c\int_M \varphi_j \varphi^j dV = -\frac{1}{n+1}\int_M (\nabla_j \Delta\varphi)\varphi^j dV = \frac{1}{n+1}\int_M (\Delta\varphi)^2 dV,$$

where $\Delta = g^{ji}\nabla_j\nabla_i$, $\varphi^j = g^{ji}\varphi_i$ and dV denotes the volume element of M. Since φ is a non-constant function, two inequalities

$$\int_M \varphi_j\varphi^j dV > 0, \quad \int_M (\Delta\varphi)^2 dV > 0$$

hold and consequently c is necessarily positive.

LEMMA 2 (Yano and Hiramatu [9]). *If a complete and simply connected Kaehlerian manifold M with positive constant scalar curvature K admits a non-affine H-projective vector field v^h and if the vector field w^h defined by (2.50) is a Killing vector field, then M is isometric to a complex projective space CP^n with Fubini-Study metric of constant holomorphic sectional curvature $\dfrac{K}{n(n+1)}$.*

PROOF. We have, from (1.1),

(3.2) $$\nabla_j(\nabla_i v_h + \nabla_h v_i) = 2\rho_j g_{ih} + \rho_i g_{jh} + \rho_h g_{ji} - F_{ji}F_h{}^t\rho_t - F_{jh}F_i{}^t\rho_t.$$

If w^h is a Killing vector field then

$$\nabla_i w_h + \nabla_h w_i = 0$$

holds and consequently

$$2(n+1)\nabla_i\rho_h + \frac{K}{2n}(\nabla_i v_h + \nabla_h v_i) = 0,$$

from which and (3.2), we find

$$\nabla_j\nabla_i\rho_h + \frac{K}{4n(n+1)}(2\rho_j g_{ih} + \rho_i g_{jh} + \rho_h g_{ji} - F_{ji}F_h{}^t\rho_t - F_{jh}F_i{}^t\rho_t) = 0.$$

Thus the lemma follows from Theorem B.

REMARK. Using Lemma 1, we see that if, in Lemma 2, M is compact then we can remove the positiveness of the scalar curvature K from the assumption.

LEMMA 3 (Yano and Hiramatu [9]). *For an H-projective vector field v^h on a complex $n > 1$ dimensional compact Kaehlerian manifold M with constant scalar curvature K, we have*

(3.3) $$\int_M (\nabla_t w^t)^2 dV = \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dV.$$

PROOF. By using a well known formula [7, 8] on a compact orientable Riemannian manifold, we have

$$\int_M (\nabla^i\nabla_i w^h + K_i{}^h w^i)w_h dV - \int_M (\nabla_t w^t)^2 dV + \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dV = 0.$$

On the other hand, as was stated in Section 2, the associated vector field ρ^h is contravariant analytic and hence satisfies

$$\nabla^i \nabla_i \rho^h + K_i{}^h \rho^i = 0 .$$

Consequently (3.3) follows from (2.39), (2.50) and the above relations since K is a constant.

LEMMA 4 (Yano and Hiramatu [9]). *For an H-projective vector field v^h on a complex $n > 1$ dimensional compact Kaehlerian manifold M with constant scalar curvature K, we have*

$$(3.4) \qquad \int_M G_{ji} \rho^j w^i dV = \frac{1}{4(n+1)} \int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV .$$

PROOF. The associated vector field ρ^h is contravariant analytic and hence satisfies

$$\nabla^j \nabla_j \rho^i + K_j{}^i \rho^j = 0 ,$$

from which and the equality

$$\nabla_i \nabla_t \rho^t = \nabla^t \nabla_t \rho_i - K_{ji} \rho^j$$

we find

$$\nabla_i \nabla_t \rho^t = -2 K_{ji} \rho^j .$$

Using the above relation, (2.40), (2.41), (2.50) and Lemma 3, we have

$$\int_M G_{ji} \rho^j w^i dV = \int_M K_{ji} \rho^j w^i dV - \frac{K}{2n} \int_M \rho_i w^i dV$$

$$= -\frac{1}{2} \int_M (\nabla_i \nabla_t \rho^t) w^i dV - \frac{K}{2n} \int_M \rho_i w^i dV$$

$$= -\frac{1}{2} \int_M (\nabla_i \nabla_t \rho^t) w^i dV - \frac{K}{4n(n+1)} \int_M (\nabla_i \nabla_t v^t) w^i dV$$

$$= -\frac{1}{2(n+1)} \int_M \left[\nabla_i \nabla_t \left\{ (n+1) \rho^t + \frac{K}{2n} v^t \right\} \right] w^i dV$$

$$= -\frac{1}{2(n+1)} \int_M (\nabla_i \nabla_t w^t) w^i dV = \frac{1}{2(n+1)} \int_M (\nabla_t w^t)^2 dV$$

$$= \frac{1}{4(n+1)} \int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV .$$

LEMMA 5 (Yano and Hiramatu [9]). *For an H-projective vector field v^h on a complex $n > 1$ dimensional compact Kaehlerian manifold M with constant scalar curvature K, we have*

$$(3.5) \qquad \int_M (\nabla^j L_v G_{ji}) w^i dV = \frac{1}{2} \int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV .$$

PROOF. Integrating

$$\nabla^j \{(L_v G_{ji})w^i\} = (\nabla^j L_v G_{ji})w^i + (L_v G_{ji})\nabla^j w^i$$

$$= (\nabla^j L_v G_{ji})w^i + \frac{1}{2}(L_v G_{ji})(\nabla^j w^i + \nabla^i w^j)$$

over M and using (2.49), we have (3.5).

LEMMA 6 (Yano and Hiramatu [9]). *For an H-projective vector field v^h on a complex $n>1$ dimensional compact Kaehlerian manifold M with constant scalar curvature K, we have*

$$(3.6) \qquad \int_M g^{kj}(L_v \nabla_k G_{ji})w^i dV = \frac{n}{2(n+1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dV.$$

PROOF. Substituting (1.1) into

$$g^{kj}(L_v \nabla_k G_{ji})w^i = (\nabla^j L_v G_{ji})w^i - g^{kj}(L_v\{_k{}^t{}_j\})G_{ti}w^i - g^{kj}(L_v\{_k{}^t{}_i\})G_{jt}w^i,$$

we have

$$g^{kj}(L_v \nabla_k G_{ji})w^i = (\nabla^j L_v G_{ji})w^i - 2G_{ji}\rho^j w^i.$$

Integrating this over M and using Lemmas 4 and 5, we have (3.6).

LEMMA 7 (Yano and Hiramatu [10]). *For an H-projective vector field v^h on a complex $n>1$ dimensional compact Kaehlerian manifold M with constant scalar curvature K, we have*

$$(3.7) \qquad \int_M (\nabla^k L_v Z_{kji}{}^h)g^{ji}w_h dV = \frac{1}{n+1}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dV.$$

PROOF. By using (2.51), we have

$$(\nabla^k L_v Z_{kji}{}^h)g^{ji}w_h$$

$$= \frac{1}{2(n+1)}\{(\nabla^k L_v G_{ji})g^{ji}w_k - (\nabla^j L_v G_{ji})w^i$$

$$- F_k{}^h(\nabla^k L_v G_{jt})F_i{}^t g^{ji}w_h + F_j{}^h(\nabla^k L_v G_{kt})F_i{}^t g^{ji}w_h$$

$$- F_k{}^t(\nabla^k L_v G_{jt})F_i{}^h g^{ji}w_h + F_j{}^t(\nabla^k L_v G_{kt})F_i{}^h g^{ji}w_h\}.$$

Here we notice that

$$-F_k{}^h(\nabla^k L_v G_{jt})F_i{}^t g^{ji}w_h = -F_k{}^h(\nabla^k L_v G_{jt})F^{jt}w_h = 0,$$

$$F_j{}^h(\nabla^k L_v G_{kt})F_i{}^t g^{ji}w_h = (\nabla^j L_v G_{ji})w^i,$$

$$-F_k{}^t(\nabla^k L_v G_{jt})F_i{}^h g^{ji}w_h = F_j{}^t(\nabla^k L_v G_{tk})F_i{}^h g^{ji}w_h = (\nabla^j L_v G_{ji})w^i$$

and

$$F_j{}^t(\nabla^k L_v G_{kt})F_i{}^h g^{ji}w_h = (\nabla^j L_v G_{ji})w^i$$

hold. Therefore we have

$$(\nabla^k L_v Z_{kji}{}^h)g^{ji}w_h = \frac{1}{2(n+1)}\{(\nabla^k L_v G_{ji})g^{ji}w_k + 2(\nabla^j L_v G_{ji})w^i\}.$$

Integrating this over M and using (2.49) and Lemmas 3 and 5, we find

$$\int_M (\nabla^k L_v Z_{kji}{}^h)g^{ji}w_h dV = \frac{1}{2(n+1)}\left\{\int_M (\nabla^k L_v G_{ji})g^{ji}w_k dV + 2\int_M (\nabla^j L_v G_{ji})w^i dV\right\}$$

$$= \frac{1}{2(n+1)}\left\{-\int_M (L_v G_{ji})g^{ji}\nabla_t w^t dV + 2\int_M (\nabla^j L_v G_{ji})w^i dV\right\}$$

$$= \frac{1}{n+1}\left\{\int_M (\nabla_t w^t)^2 dV + \int_M (\nabla^j L_v G_{ji})w^i dV\right\}$$

$$= \frac{1}{n+1}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dV.$$

LEMMA 8 (Yano and Hiramatu [10]). *For an H-projective vector field* v^h *on a complex* $n>1$ *dimensional compact Kaehlerian manifold* M *with constant scalar curvature* K, *we have*

$$(3.8)\quad \int_M (\nabla^k L_v Z_{kjih})g^{ji}w^h dV = \frac{3}{2(n+1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dV.$$

PROOF. By using $Z_{kji}{}^h g^{ji} = G_k{}^h$ and

$$(\nabla^k Z_{kji}{}^h)g^{ji} = \nabla^k G_k{}^h = \frac{n-1}{2n}\nabla^h K = 0,$$

we have

$$(\nabla^k L_v Z_{kjih})g^{ji}w^h = \{\nabla^k L_v(Z_{kji}{}^t g_{th})\}g^{ji}w^h$$

$$= (\nabla^k L_v Z_{kji}{}^h)g^{ji}w_h + G_k{}^t(\nabla^k L_v g_{th})w^h.$$

Substituting (2.43) into this, we find

$$(\nabla^k L_v Z_{kjih})g^{ji}w^h = (\nabla^k L_v Z_{kji}{}^h)g^{ji}w_h + 2G_{ji}\rho^j w^i,$$

and consequently, integrating this over M and using Lemmas 4 and 7, we have (3.8).

Now we prove Theorem A. Using

$$\nabla^k Z_{kjih} = \nabla_k Z_{hij}{}^k$$

and

$$(L_v Z_{kjih})g^{ji} = L_v G_{kh} - Z_{kjih}L_v g^{ji},$$

we have

$$(\nabla^k L_v Z_{kjih})g^{ji}w^h = (\nabla^k L_v G_{kh})w^h - (\nabla_k Z_{hij}{}^k)(L_v g^{ji})w^h - Z_{kjih}(\nabla^k L_v g^{ji})w^h.$$

Substituting (2.33) and (2.45) into this, we find

$$(\nabla^k L_v Z_{kjih})g^{ji}w^h = (\nabla^j L_v G_{ji})w^i - (\nabla_h G_{ij})(L_v g^{ji})w^h$$
$$+ (\nabla_i G_{hj})(L_v g^{ji})w^h + 4G_{ji}\rho^j w^i,$$

from which, integrating over M,

$$\int_M (\nabla^k L_v Z_{kjih})g^{ji}w^h dV = \int_M (\nabla^j L_v G_{ji})w^i dV - \int_M (\nabla_h G_{ji})(L_v g^{ji})w^h dV$$
$$+ \int_M (\nabla_j G_{ih})(L_v g^{ji})w^h dV + 4\int_M G_{ji}\rho^j w^i dV.$$

Here we notice that we have, using (2.44) and (2.49),

$$-\int_M (\nabla_h G_{ji})(L_v g^{ji})w^h dV = \int_M G_{ji}(\nabla_h L_v g^{ji})w^h dV + \int_M G_{ji}(L_v g^{ji})\nabla_t w^t dV$$
$$= -4\int_M G_{ji}\rho^j w^i dV - \int_M (L_v G_{ji})g^{ji}\nabla_t w^t dV$$
$$= -4\int_M G_{ji}\rho^j w^i dV + 2\int_M (\nabla_t w^t)^2 dV$$

and, using

$$(\nabla_j G_{ih})g^{ji} = \frac{n-1}{2n}\nabla_h K = 0,$$

$$\int_M (\nabla_j G_{ih})(L_v g^{ji})w^h dV = -\int_M g^{ji}(L_v \nabla_j G_{ih})w^h dV.$$

Consequently, we have

$$\int_M (\nabla^k L_v Z_{kjih})g^{ji}w^h dV$$
$$= 2\int_M (\nabla_t w^t)^2 dV + \int_M (\nabla^j L_v G_{ji})w^i dV - \int_M g^{kj}(L_v \nabla_k G_{ji})w^i dV,$$

from which, using Lemmas 3, 5 and 6,

$$\int_M (\nabla^k L_v Z_{kjih})g^{ji}w^h dV = \frac{2n+3}{2(n+1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dV.$$

From this and Lemma 8, we have

$$\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dV = 0,$$

from which

$$\nabla_j w_i + \nabla_i w_j = 0,$$

that is, the vector field w^h is a Killing vector field. Thus, Theorem A follows from Lemma 2.

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