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ON A CHARACTERIZATION OF QUATERNION PROJECTIVE SPACE BY DIFFERENTIAL EQUATIONS

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§ 1. Introduction.

The existence of a non-trivial solution of certain differential equations on a Riemannian manifold M often determines some geometric and topological properties of M. For example, in [9] Obata proved the following Theorems 1 and 2.

THEOREM 1. *Let M be a complete connected and simply connected Riemannian manifold of dimension $n(\geqq 2)$. In order for M to admit a non-trivial solution f for the system of differential equations*

$$\text{(I)} \qquad \nabla_k \nabla_j f_i + k(2f_k g_{ji} + f_j g_{ki} + f_i g_{kj}) = 0\,, \qquad k = \text{const} > 0\,,$$

where $f_i = \nabla_k f$, it is necessary and sufficient that M be isometirc with a sphere S^n of radius $1/\sqrt{k}$ in the Euclidean $(n+1)$-space.

THEOREM 2. *Let M be a complete connected and simply connected Kaehler manifold of dimension $2m(\geqq 4)$. In order for M to admit a non-trivial solution f for the system of differential equations*

$$\text{(II)} \qquad \nabla_k \nabla_j f_i + k(2f_k g_{ji} + f_j g_{ki} + f_i g_{kj} - F_j{}^t f_t F_{ki} - F_i{}^t f_t F_{kj}) = 0\,, \quad k = \text{const} > 0\,,$$

where $F_j{}^i$ is the complex structure of M, it is necessary and sufficient that M be isometric with the complex projective space $P^m(C)$ with Fubini-study metric of constant holomorphic sectional curvature $4k$.

In [1], Blair showed a relation between Theorems 1 and 2 by deducing Theorem 2 from Theorem 1 in the case where M is a Hodge manifold. The idea of his proof is to show that the projection of (I) on S^{2m+1} via the Hopf-fibration $\pi: S^{2m+1} \to P^m(C)$ gives the equation (II) on $P^m(C)$. In a similar way we can characterize the quaternion projective space $P^m(H)$ by differential equations via the Hopf-fibration $\tilde{\pi}: S^{4m+3} \to P^m(H)$. The purpose of this parer is to prove the following Theorem 3.

THEOREM 3. *Let M be a complete connected quaternion Kaehler manifold of dimension $4m(\geqq 8)$. In order for M to admit a non-trivial solution f for the*

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system of differentirl equations

(III) $$\nabla_c\nabla_b f_a+k(2f_c g_{ba}+f_b g_{ca}+f_a g_{cb}-\Lambda_{cba}{}^e f_e-\Lambda_{cab}{}^e f_e)=0\,,\quad k=\text{const}>0\,,$$

where $\Lambda_{cba}{}^e$ *is a global tensor field on* M *defined by* (2.3), *it is necessary and sufficient that* M *be ısometric wıth the quaternion projective space* $P^m(H)$ *with constant* Q*-sectional curvature* $4k$.

We remark that grad f in Theorem 1 and 2 are an infinitesimal projective transformation and an infinitesimal H-projective transformation respectively. From our case, as an analogue, we can expect that grad f in Theorem 3 gives a certain special infinitesimal transformation. Namely, in a quaternion Kaehler space, the one parameter group generated by grad f, where f is a non-trivial solution of (III), leaves the family of all curves r whose covariant derivative of the tangent vector field $\dot{r}$ of r is contained in the quaternion subspace spanned by r.

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§ 2. Quaternion Kaehler manifolds (See [2].).

Let M be a differentiable manifold of dimension n and there exist a subbundle V of the tensor bundle of type (1, 1) over M satisfying the following condition:

(a) In any coordinate neighborhood U of M, there is a local basis $\{F, G, H\}$ of the bundle V, where $\{F, G, H\}$ are tensor fields of type (1, 1) in U satisfying

(2.1) $$F^2=G^2=H^2=-I\,,$$
$$GH=-HG=F\,,\quad HF=-FH=G\,,\quad FG=-GF=H\,,$$

I being the identity tensor field of type (1, 1) in M. Such a local basis $\{F, G, H\}$ of V is called a *canonical local basis* of V in U.

Thus the bundle V is a 3-dimensional vector bundle. Such a bundle V is called an *almost quaterion structure* and the pair (M, V) an *almost quaternion manifold.* An almost quaternion manifold is orientable and of dimension $n=4m(m\geqq 1)$.

For an almost quaternion manifold (M, V), let $\{F, G, H\}$ and $\{F', G', H'\}$ be canonical local bases of V in U and in another coordinate neighborhood U' of M, respectively. Then we have in $U\cap U'$

(2.2) $$\begin{aligned}F'&=s_{11}F+s_{12}G+s_{13}H\,,\\ G'&=s_{21}F+s_{22}G+s_{23}H\,,\\ H'&=s_{31}F+s_{32}G+s_{33}H\,,\end{aligned}$$

where $S=(s_{\alpha\beta})\in SO(3)$, $(\alpha, \beta=1, 2, 3)$, because $\{F, G, H\}$ and $\{F', G', H'\}$ satisfy (2.1). Thus, if on U we can put Λ as following:

$$\Lambda=F\otimes F+G\otimes G+H\otimes H, \tag{2.3}$$

then using (2.2) gives that Λ determines in M a global tensor field of type (2, 2), which will be denoted also by Λ.

Next let there be given an almost quaternion structure V in a Riemannian manifold (M, g) and assume that for any canonical local basis $\{F, G, H\}$ of V, each of F, G and H is almost Hermitian with respect to g. Moreover we suppose that the set (M, g, V) satisfies the following condition:

(b) If ϕ is a cross-section of the bundle V, then $\nabla_X\phi$ is also a cross-section of V for any vector field X on M, where ∇ denotes the Riemannian connection of the Riemannian manifold (M, g, V).

Such a set (M, g, V) is called a *quaternion Kaehler manifold* and the set $\{g, V\}$ a quaternion Kaehler structure in M. The condition (b) is equivalent to the following condition:

(b′) For a canonical local basis $\{F, G, H\}$ of V in U,

$$\begin{aligned} \nabla_X F&=r(X)G-q(X)H, \\ \nabla_X G&=-r(X)F+p(X)H, \\ \nabla_X H&=q(X)F-p(X)G \end{aligned} \tag{2.4}$$

for any vector field X on M, where p, q and r are local 1-forms in U. Thus, using (2.4), we easily find

$$\nabla\Lambda=0. \tag{2.5}$$

Here, we can easily verify that condition (2.5) is equivalent to condition (b′).

It is known that any quaternion Kaehler manifold is an Einstein space, i.e., that the Ricci tensor S of (M, g) has the form

$$S=\frac{s}{4m}I, \tag{2.6}$$

s being the scalar curvature of (M, g) which is a constant if M is connected, where $\dim M=4m$ $(m\geqq 2)$.

We denote by $R_{dcb}{}^e$ components of the curvature tensor of (M, g) and put $R_{dcba}=R_{dcb}{}^e g_{ea}$. Put $F_{ba}=F_b{}^e g_{ea}$, $G_{ba}=G_b{}^e g_{ea}$, $H_{ba}=H_b{}^e g_{ea}$, which are all skew-symmetric.

Let a function f satisfy the differential equation (III). Then using (III) and the Ricci identity we get.

$$\begin{aligned} \nabla_c\nabla_b f_a-\nabla_b\nabla_c f_a&=-R_{cba}{}^e f_e \\ &=k(f_b g_{ca}-f_c g_{ba}+2\Lambda_{cba}{}^e f_e+\Lambda_{cab}{}^e f_e+\Lambda_{bac}{}^e f_e). \end{aligned}$$

Contracting with g^{ba} we have from (2.6)

$$(s-4m(4m+8)k)f_c=0\,,$$

where the function f is non-trivial. Then we have

LEMMA 2.2. *If M admits a non-trivial solution for* (III), *then the scalar curvature s is equal to* $4m(4m+8)k>0$.

Next the following integral formula is known:

PROPOSITION 2.3 (Ishihara [3].). *Let M be a compact quaternion Kaehler manifold. Then*

$$\int_M (3m(\nabla^a\nabla_a X^b+s/(4m+8)\cdot X^b))X_b+1/16\|\mathcal{L}_X\Lambda\|^2 \\ +(F_a{}^b\nabla_b X^a)^2+(G_a{}^b\nabla_b X^a)+(H_a{}^b\nabla_b X^a)^2)=0\,,$$

where X^b is a vector field on M.

Assume that M admits a non-trivial solution f for (III). Contracting (III) with g^{cb} and using Lemma 2.2, we have

$$\nabla_a\nabla^a f_b+s/(4m+8)\cdot f_b=0\,. \tag{2.7}$$

Because of skew-symmetry of F, G and H, we get

$$F_c{}^b\nabla_b f^c=G_c{}^b\nabla_b f^c=H_c{}^b\nabla_b f^c=0\,. \tag{2.8}$$

M is an Einstein space whose scalar curvature is positive, because of Lemma 2.2. Thus M is compact, since M is complete. Substituting X^a by f^a in Proposition 2.3 and making use of (2.7) and (2.8), we get

$$\mathcal{L}_{\mathrm{grad}\,f}\Lambda=0\,. \tag{2.9}$$

From (2.9) and (2.5), we have easily in the coordinate neignborhood U

$$\begin{aligned} &\nabla_a f_c F_b{}^c+\nabla_b f_c F_a{}^c=0\,,\\ &\nabla_a f_c G_b{}^c+\nabla_b f_c G_a{}^c=0\,,\\ &\nabla_a f_c H_b{}^c+\nabla_b f_c H_a{}^c=0\,. \end{aligned} \tag{2.10}$$

§ 3. Fibred space with Sasakian 3-structure (See [4].).

Let $\tilde{M}$ have a Sasakian 3-structure and M be a quaternion Kaehler manifold, and assume that there exists a fibration $\pi:\tilde{M}\to M$ (See [5].). In such a case, $\tilde{M}$ is necessarily of dimension $n+3=4m+3$. We now assume that $\dim M>7$ (i. e. $m>1$). The fundamental geometry in such a situation has already been discussed in [4] and [5]. We shall recall some notions and results given in [4] and [5].

We take coordinate neighborhoods $\{\tilde{U}, x^h\}$ of $\tilde{M}$ such that $\pi(\tilde{U})=U$ are coordinate neighborhood of M with local coordinates (v^a). Then the projection $\pi: \tilde{M} \to M$ may be expressed with respect to $\{\tilde{U}, x^h\}$ and $\{U, v^a\}$ by certain equations of the form

$$v^a = v^a(x^1, \cdots, x^{n+3}),$$

v^a denoting coordinates in U of the projection $P = \pi(\sigma)$ of a point σ with coordinates x^h in $\tilde{U}$, where $v^a(x^1, \cdots, x^{n+3})$ are differentiable functions of variables x^h with Jacobian matrix $(\partial v^a/\partial x^h)$ of the maximal rank $4m$. We take a fibre F such that $F \cap \tilde{U} \neq \emptyset$. Then, we may assume that $F \cap \tilde{U}$ is connected. We can introduce local coordinates (u^α) in $F \cap \tilde{U}$ in such a way that (v^a, u^α) is a system of local coordinates in $\tilde{U}$, (v^a) being coordinates of $\pi(F)$ in U.

We now put $E_i{}^a = \partial v^a/\partial x^i$ and $C_\alpha = \partial/\partial u^\alpha$. Denoting by $C^h{}_\alpha$ components of C_α in U, we put $C_i{}^\alpha = \tilde{g}_{ih}\bar{g}^{\alpha\beta}C^h{}_\beta$, where $\tilde{g}_{ji}$ are components of $\tilde{g}$ in $\tilde{U}$, $\bar{g}_{\alpha\beta} = g_{ji}C^j{}_\alpha C^i{}_\beta$ and $(\bar{g}^{\alpha\beta}) = (\bar{g}_{\alpha\beta})^{-1}$. We next define E^h by $(E^h{}_a, C^h{}_\alpha) = (E_i{}^a, C_i{}^\alpha)^{-1}$. We now define three tensor fields ϕ, ψ and θ of type (1, 1) by

$$\phi = \tilde{\nabla}\xi, \qquad \psi = \tilde{\nabla}\eta, \qquad \theta = \tilde{\nabla}\zeta.$$

Then we can put in U, denoting E^b and E_a a vector field and a 1-form whose components are $E_i{}^b$ and $E^i{}_a$ respectively,

$$\phi^H = \phi_b{}^a E^b \otimes E_a, \qquad \psi^H = \psi_b{}^a E^b \otimes E_a, \qquad \theta^H = \theta_b{}^a E^b \otimes E_a, \tag{3.3}$$

where ϕ^H denotes the horizontal part of ϕ and so forth, $\phi_b{}^a$, $\psi_b{}^a$, $\theta_b{}^a$ being local functions in U and ϕ^H, ψ^H, θ^H satisfy (2.1) (See [5].). We easily have

$$\begin{aligned} &\phi_{ba} = -\phi_{ab} = \phi_b{}^e g_{ea}, \qquad \psi_{ba} = -\psi_{ab} = \psi_b{}^e g_{ea}, \\ &\theta_{ba} = -\theta_{ab} = \theta_b{}^e g_{ea}, \end{aligned} \tag{3.4}$$

where $g_{ab} = g_{ji}E^j{}_a E^i{}_b$ which is a Riemannian metric of M. We get the Co-Gauss formulas (See [5], [6].)

$$\begin{aligned} &\tilde{\nabla}_j E_i{}^a = -\left\{{a \atop cb}\right\} E_j{}^c E_i{}^b + h_b{}^a{}_\beta (E_j{}^b C_i{}^\beta + C_j{}^\beta E_i{}^b), \\ &\tilde{\nabla}_j C_i{}^\alpha = -h_{cb}{}^\alpha E_j{}^c E_i{}^b - P_{c\beta}{}^\alpha E_j{}^c C_i{}^\beta - \left\{{\alpha \atop \beta\gamma}\right\} C_j{}^\beta C_i{}^\gamma, \end{aligned} \tag{3.5}$$

where $h_b{}^a{}_\beta$, $P_{c\beta}{}^\alpha$, $\left\{{a \atop cb}\right\}$ and $\left\{{\alpha \atop \beta\gamma}\right\}$ are local functions defined in $\tilde{U}$ respectively. In particular, $\left\{{a \atop cb}\right\}$ and $\left\{{\alpha \atop \beta\gamma}\right\}$ are Christoffel's symbols formed with g_{ab}, and $\bar{g}_{\alpha\beta}$ respectively. Furthermore we get

$$h_b{}^a{}_\beta = -(a_\beta \phi_b{}^a + b_\beta \psi_b{}^a + c_\beta \theta_b{}^a), \tag{3.6}$$

where we put $\xi = a^\alpha C_\alpha$, $\eta = b^\alpha C_\alpha$, $\zeta = c^\alpha C_\alpha$ and $a_\beta = \bar{g}_{\beta\alpha}a^\alpha$, $b_\beta = \bar{g}_{\beta\alpha}b^\alpha$, $c_\beta = \bar{g}_{\beta\alpha}c^\alpha$ in $\tilde{U}$.

The following structure equation for π is satisfied (See [6], Chapter I, 6.):

$$K_{kji}{}^h E^k{}_d E^j{}_c E^i{}_b C_h{}^\alpha = {}'\nabla_d h_{cb}{}^\alpha - {}'\nabla_c h_{db}{}^\alpha, \tag{3.7}$$

$$K_{kji}{}^{h}E^{k}{}_{d}C^{j}{}_{\beta}E^{i}{}_{b}C_{h}{}^{\alpha}=-{}''\nabla_{\beta}h_{db}{}^{\alpha}+h_{d}{}^{e}{}_{\beta}h_{eb}{}^{\alpha}\,, \tag{3.8}$$

$K_{kji}{}^{h}$ being curvature tensor of $\tilde{M}$ and

$$'\nabla_{d}h_{cb}{}^{\alpha}=\partial_{d}h_{cb}-\left\{{e \atop dc}\right\}h_{eb}{}^{\alpha}-\left\{{e \atop db}\right\}h_{ce}{}^{\alpha}+P_{d\varepsilon}{}^{\alpha}h_{cb}{}^{\varepsilon}\,,$$

$$''\nabla_{\beta}h_{db}{}^{\alpha}=\partial_{\beta}h_{db}{}^{\alpha}+\left\{{\alpha \atop \beta\gamma}\right\}h_{bd}{}^{\gamma}-h_{d}{}^{e}{}_{\beta}h_{eb}{}^{\alpha}-h_{b}{}^{e}{}_{\beta}h_{de}{}^{\alpha}\,.$$

Using the Ricci identity for ξ, η, ζ and (3.6), (3.7), (3.8), we have

$$\partial_{\beta}h_{a}{}^{e}{}_{\alpha}f_{e}-\left\{{\gamma \atop \alpha\beta}\right\}h_{a}{}^{e}{}_{\gamma}f_{e}+f_{e}h_{d}{}^{e}{}_{\alpha}h_{a}{}^{d}{}_{\beta}+f_{a}\bar{g}_{\alpha\beta}=0\,, \tag{3.9}$$

$$\partial_{c}h_{a}{}^{e}{}_{\alpha}f_{e}+\left\{{d \atop ec}\right\}f_{d}h_{a}{}^{e}{}_{\alpha}-\left\{{d \atop ca}\right\}f_{e}h_{d}{}^{e}{}_{\alpha}-P_{c\alpha}{}^{\beta}h_{a}{}^{e}{}_{\beta}f_{e}=0\,, \tag{3.10}$$

and

$$f_{e}h_{d}{}^{e}{}_{\alpha}h_{c}{}^{d}{}_{\beta}+f_{e}h_{d}{}^{e}{}_{\beta}h_{c}{}^{d}{}_{\alpha}+2f_{c}\bar{g}_{\alpha\beta}=0\,. \tag{3.11}$$

Let f be a function on M. We now consider a tensor L_{kji} given by

$$L_{kji}=\tilde{\nabla}_{k}\tilde{\nabla}_{j}\tilde{\nabla}_{i}\hat{f}+2\tilde{f}_{k}\tilde{g}_{ji}+\tilde{f}_{j}\tilde{g}_{ki}+\tilde{f}_{i}\tilde{g}_{kj}\,,$$

where $\hat{f}$ denotes the lift of f (i. e., $\tilde{f}(\sigma)=f\circ\pi(\sigma)$.). Now we have

$$\tilde{\nabla}_{i}\tilde{f}=\tilde{f}_{i}=E_{i}{}^{a}\nabla_{a}f\,,$$

in $\tilde{U}$, ∇_{a} being a formal covariant derivative with respect to $\left\{{a \atop cb}\right\}$. Using (3.15), we get

$$\tilde{\nabla}_{j}\tilde{\nabla}_{i}\tilde{f}=(\nabla_{a}f_{b})E_{j}{}^{a}E_{i}{}^{b}+f_{a}h_{b}{}^{a}{}_{\alpha}(E_{j}{}^{b}C_{i}{}^{\alpha}+C_{j}{}^{\alpha}E_{i}{}^{b})\,. \tag{3.12}$$

Moreover differentiating (3.12) covariantly and using (3.15), we have

$$\begin{aligned}\tilde{\nabla}_{k}\tilde{\nabla}_{j}\tilde{\nabla}_{i}\tilde{f}=&(\nabla_{c}\nabla_{b}f_{a}-h_{b}{}^{e}{}_{\alpha}f_{e}h_{ca}{}^{\alpha}-h_{a}{}^{e}{}_{\alpha}f_{e}h_{cb}{}^{\alpha})E_{k}{}^{c}E_{j}{}^{b}E_{i}{}^{a}\\&+(\nabla_{a}f_{d}h_{b}{}^{d}{}_{\alpha}+\nabla_{b}f_{d}h_{a}{}^{b}{}_{\alpha})C_{k}{}^{\alpha}E_{j}{}^{b}E_{i}{}^{a}\\&+W_{ca\alpha}E_{k}{}^{c}E_{i}{}^{a}C_{j}{}^{\alpha}+W_{cb\alpha}E_{k}{}^{c}E_{j}{}^{b}C_{i}{}^{\alpha}\\&+Z_{a\beta\alpha}C_{k}{}^{\beta}C_{j}{}^{\alpha}E_{i}{}^{a}+Z_{b\beta\alpha}C_{k}{}^{\beta}C_{i}{}^{\alpha}E_{j}{}^{b}\\&+(f_{e}h_{d}{}^{e}{}_{\alpha}h_{c}{}^{d}{}_{\beta}+f_{e}h_{d}{}^{e}{}_{\beta}h_{c}{}^{d}{}_{\alpha})E_{k}{}^{c}C_{j}{}^{\beta}C_{i}{}^{\alpha}\,,\end{aligned} \tag{3.13}$$

where $W_{ca\alpha}$ and $Z_{a\beta\alpha}$ are defined respectively by

$$\begin{aligned}W_{ca\alpha}=&\nabla_{a}f_{d}h_{c}{}^{d}{}_{\alpha}+\nabla_{c}f_{d}h_{a}{}^{d}{}_{\alpha}+\nabla_{c}h_{a}{}^{e}{}_{\alpha}f_{e}+\left\{{d \atop ec}\right\}f_{d}h_{a}{}^{e}{}_{\alpha}\\&-\left\{{d \atop ca}\right\}f_{e}h_{d}{}^{e}{}_{\alpha}-P_{c\alpha}{}^{\beta}h_{a}{}^{d}{}_{\beta}f_{d}\,,\end{aligned}$$

$$Z_{\alpha\beta\alpha}=\partial_\beta h_a{}^e{}_\alpha f_e-\left\{{\gamma \atop \alpha\beta}\right\}f_e h_a{}^e{}_\gamma+f_e h_d{}^e{}_\alpha h_a{}^d{}_\beta .$$

Thus, substituting (3.13) in L_{kji} from (3.9), (3.10) and (3.11), we have

$$\begin{aligned} L_{kji}=&(\nabla_c\nabla_b f_a-h_b{}^e{}_\alpha h_{ca}{}^\alpha f_e-h_a{}^e{}_\alpha h_{cb}{}^\alpha f_e+2f_c g_{ba}+f_b g_{ca}+f_a g_{cb})E_k{}^c E_j{}^b E_i{}^a \\ &+(\nabla_a f_d h_b{}^d{}_\alpha+\nabla_b f_d h_a{}^d{}_\alpha)C_k{}^\alpha E_j{}^b E_i{}^a \\ &+(\nabla_a f_d h_c{}^d{}_\alpha+\nabla_c f_d h_a{}^d{}_\alpha)E_k{}^c C_j{}^\alpha E_i{}^a \\ &+(\nabla_b f_d h_c{}^d{}_\alpha+\nabla_c f_d h_b{}^d{}_\alpha)E_k{}^c E_j{}^b C_i{}^\alpha . \end{aligned} \tag{3.14}$$

§ 4. The construction of Hopf-fibration from the Sasakian 3-structure.

In this section we construct the Hopf-fibration $S^3\to S^{4m+3}\to P^m(H)$ by using the given Sasakian 3-structure on the sphere S^{4m+3}. The construction of the Hopf-fibration $S^1\to S^{2m+1}\to P^m(C)$ is studied by Yano and Ishihara [11].

First suppose that $\imath: S^{4m+3}(1)\to R^{4m+4}$ is an imbedding given by the equation $\sum_{A-1}^{4m+4} y_A{}^2=1$. Setting $y_\imath=u_\imath$, $(1\leqq i\leqq 4m+3)$, we get $y_{4m+4}=\pm[1-\sum_{\imath=1}^{4m+3}u_\imath{}^2]^{1/2}$. Then the differential i_* of the imbedding is given by

$$(\imath_*)_i{}^A=(\partial y_A/\partial u_\imath)=\begin{cases} \delta_i{}^j , & (A=j=1,\cdots,4m+3) \\ -\dfrac{u_\imath}{\lambda} , & (A=4m+4) , \end{cases}$$

where we have set $[1-\sum_{\imath=1}^{4m+3}u_\imath{}^2]^{1/2}=\lambda$ (resp. $=-\lambda$) for the hemisphere $y_{4m+4}>0$ (resp. for $y_{4m+4}<0$). The induced metric g is given by $g_{j\imath}=\delta_{ji}+u_i u_j/\lambda^2$. We take the outer normal vector N, i.e., the components N^A of N is y_A. Let v^a denote the components of vector field on R^{4m+4}. Then the components of its projection on S^{4m+3} are $v^\imath=\sum_{A=1}^{4m+4}v^A(i_*)^\imath{}_A$, where

$$(i_*)^\imath{}_A=\begin{cases} \delta_j{}^\imath-u_i u_j , & A=j , \\ -u_\imath , & A=4m+4 . \end{cases}$$

We denote $\left\{{k \atop ji}\right\}$ the Christoffel's symbol formed with g. Then we have

$$\left\{{k \atop ji}\right\}=u_k(\delta_{ji}+u_j u_\imath) . \tag{4.1}$$

Since the imbedding is totally umbilical whose principal curvature is equal to 1, we get

$$\tilde\nabla_j(i_*)_i{}^A=g_{ji}N^A , \qquad \tilde\nabla_j N^A=-(i_*)_j{}^A ,$$

where $\tilde\nabla_j$ is the van der Waerden-Bortolloti covariant derivative. Let $\xi=\sum_{\imath=1}^{4m+3}\xi^i\partial/\partial u_\imath$ be a Sasakian strncture on S^{4m+3}. We define a vector field $\tilde\xi=$

$\sum_{A=1}^{4m+4}\xi^A\partial/\partial y_A$ by $\xi^A=\sum_{i=1}^{4m+3}\xi^i(i_*)_i{}^A$. From (3.1) and (4.2), we have

$$\tilde{\nabla}_j\tilde{\nabla}_i\xi^A=-g_{ji}\xi^A\,.$$

By Obata [9], the function ξ^A can be written on S^{4m+3} as

$$\xi^A=\sum_{B=1}^{4m+4}a_{AB}y_B\,, \tag{4.3}$$

where $A=(a_{AB})$ is a constant matrix. Extending ξ^A to R^{4m+4} by homotheties centered at the origin, we can make it a vector field $\tilde{\xi}$ on the R^{4m+4} and denote it by the same letter. In fact, the components of $\tilde{\xi}$ has the same form as given by (4.3). Because of the above constructure, $\tilde{\xi}$ is orthogonal to the Normal vector N:

$$\sum_{A,B=1}^{4m+4}a_{AB}y_Ay_B=\sum_{A=1}^{4m+4}\xi^AN^A=0\,.$$

from which we have

$$a_{AB}+a_{BA}=0\,. \tag{4.4}$$

Since ξ is unit vector, $\sum_{A=1}^{4m+4}\xi^A\xi^A=1$ on S^{4m+3}. In particular at $u_i=0$, $(1\leqq i\leqq 4m+3)$, we have

$$\sum_{i=1}^{4m+3}a_{i4m+4}^2=1\,. \tag{4.5}$$

On the other hand

$$\begin{aligned}\xi^i&=\sum_{A=1}^{4m+4}\xi^A(i_*)^i{}_A\\&=\sum_{j=1}^{4m+3}(\sum a_{jB}y_B)(\delta_j{}^i-u_ju_i)+(\sum_{B=1}^{4m+4}a_{4m+4B}y_B)(-\lambda u_i)\,.\end{aligned}$$

Using (4.4), we get

$$\xi^i=\sum_{j=1}^{4m+3}a_{ij}u_j+\lambda a_{i4m+4}\,. \tag{4.6}$$

Differentiating (4.6) covariantly and using (4.1) and (4.4), we have

$$\begin{aligned}\phi_k{}^i=\nabla_k\xi^i&=\frac{\partial\xi^i}{\partial u_k}+\sum_{t=1}^{4m+3}\left\{{i\atop k\,t}\right\}\xi^t\\&=a_{ik}-\frac{1}{\lambda}a^i{}_{4m+4}+\sum_t u_i\Big(\delta_{kt}+\frac{1}{\lambda^2}u_ku_t\Big)(\sum_j a_{tj}u_j+\lambda a_{t4m+4})\\&=a_{ik}-\frac{1}{\lambda}a_{i4m+4}+(\sum_j a_{kj}u_j)u_i+\lambda a_{k4m+4}u_i+\frac{1}{\lambda}(\sum_t a_{t4m+4}u_t)u_ku_i\,.\end{aligned}$$

Then we have from (4.4)

$$\phi_{kj}=\sum_{i=1}^{4m+3}g_{ji}\phi_k{}^i=a_{jk}+\frac{1}{\lambda}a_{k4m+4}u_j-\frac{1}{\lambda}a_{j4m+4}u_k\,.$$

Since $\sum_{j=1}^{4m+3} \phi_{kj}\xi^j=0$, we get by (4.6)

$$\sum_{i,j=1}^{4m+3} a_{ik}a_{ij}u_j-\frac{1}{\lambda}\left(\sum_{i,j=1}^{4m+3} a_{ij}a_{i4m+4}u_j\right)u_k \tag{4.7}$$

$$+\lambda\left(\sum_{i=1}^{4m+3} a_{ik}a_{i4m+4}\right)-\sum_i(a_{i4m+4})^2u_k+a_{k4m+4}\left(\sum_{i=1}^{4m+3} a_{j4m+4}\right)=0\,.$$

At $u_i=0$, $(1\leqq i\leqq 4m+3)$, we have

$$\sum_{i=1}^{4m+3} a_{ij}a_{i4m+4}=0\,. \tag{4.8}$$

Because of (4.5) and (4.8), (4.7) is reduced to

$$\sum_{i=1}^{4m+3} a_{ki}a_{ij}+a_{k4m+4}a_{4m+4j}=-\delta_{kj}\,. \tag{4.9}$$

By (4.8) and (4.9), we have $A^2=-I$, where I is the identity matrix. Let $\{\xi, \eta, \zeta\}$ be the natural Sasakian 3-structure on $S^{4m+3}(1)$. From the above construction the extended vector fields $\tilde{\xi}$, $\tilde{\eta}$ and $\tilde{\zeta}$ on R^{4m+4} can be written as

$$\tilde{\xi}=AN\,,\qquad \tilde{\eta}=BN\,,\qquad \tilde{\zeta}=CN\,,$$

where the matrices $A=(a_{AB})$, $B=(b_{AB})$, $C=(c_{AB})$ are constant and skew-symmetric. They satisfy $A^2=B^2=C^2=-I$. Since $\tilde{\xi}$, $\tilde{\eta}$ and $\tilde{\zeta}$ are mutually orthogonal,

$$\sum_{A,B,C} a_{AB}b_{AC}y_By_C=\sum_{A,B,C} a_{AB}c_{AC}y_By_C=\sum_{A,B,C} b_{AB}c_{AC}y_By_C=0\,,$$

from which we have

$$\sum_B(a_{AB}b_{BC}+a_{CB}b_{BA})=\sum_B(b_{AB}c_{BC}+b_{CB}c_{BA}) \tag{4.10}$$

$$=\sum_B(c_{AB}a_{BC}+c_{CB}a_{BA})=0\,.$$

By the definition (3.2) of the Sasakian 3-structure, we get

$$[\tilde{\xi}, \tilde{\eta}]|_{S^{4m+3}}=[i_*\xi, i_*\eta]=i_*[\xi, \eta]=2\tilde{\zeta}|_{S^{4m+3}}\,.$$

From the above equation, using (4.10), we have

$$\sum_{A,C}(a_{AC}b_{AB}-c_{CB})y_C=0\,.$$

This means that $BA=-AB=C$. Similarly, we have $CB=-BC=A$ and $AC=-CA=B$. Then $\{A, B, C\}$ defines a quaternion structure on R^{4m+4}. We consider the distribution D spanned by ξ, η and ζ. Then the projection $\pi: S^{4m+3}\to S^{4m+3}/D$ is the Hopf-fibration. So we get

PROPSSITION 4.1. *Let $S^{4m+3}(1)$ be a shere of radius 1 and have the natural Sasakian 3-structure $\{\xi, \eta, \zeta\}$. Then S^{4m+3}/D is a quaternion projective space*

$P^m(H)$.

§ 5. Proof of Theorem 3.

We first note that it suffices to prove the theorem for $k=1$. For, if $k\neq 1$, the homothetic change $\tilde{g}\rightarrow g=kg$ of metric transforms the differential equation given in Theorem 3 into the corresponding one with $k=1$. We are now going to give a proof of Theorem 3.

Sufficiency. Let $S^{4m+3}(1)$ be the sphere in R^{4m+4} with its natural Sasakian 3-structure $\{\xi, \eta, \zeta\}$. i.e., according to the notation of § 4, we may put

$$\xi=FN, \qquad \eta=GN, \qquad \zeta=HN,$$

where F, G, H are matrices defined by the following:

$$F=\left(\begin{array}{cc|c} \begin{smallmatrix}0&1\\-1&0\end{smallmatrix} & 0 & \\ 0 & \begin{smallmatrix}0&1\\-1&0\end{smallmatrix} & \\ \hline & & \ddots \end{array}\right), \quad G=\left(\begin{array}{cc|c} 0 & \begin{smallmatrix}1&0\\0&1\end{smallmatrix} & \\ \begin{smallmatrix}1&0\\0&-1\end{smallmatrix} & 0 & \\ \hline & & \ddots \end{array}\right), \quad H=\left(\begin{array}{cc|c} 0 & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & \\ \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & 0 & \\ \hline & & \ddots \end{array}\right).$$

We define a function $\tilde{f}$ on S^{4m+3} by $\tilde{f}=(1/2)(u_1^2+u_2^2+u_3^2+u_4^2)$. Then it is easily checked that $\tilde{f}$ is a solution of the differential equation (I) and $\mathcal{L}_\xi\tilde{f}=\mathcal{L}_\eta\tilde{f}=\mathcal{L}_\zeta\tilde{f}=0$. We now consider the Hopf-fibration $\pi: S^{4m+3}\rightarrow P^m(H)$. Then $\tilde{f}$ is projectable with respect to π. We can define a function f on M by $f\circ\pi=\tilde{f}$. From (3.13), we get

$$\nabla_c\nabla_b f_a-h_b{}^e{}_\alpha f_e h_{ca}{}^\alpha-h_a{}^e{}_\alpha f_e h_{cb}{}^\alpha+2f_c g_{ba}+f_b g_{ca}+f_a g_{cb}=0\,.$$

Thus, by (3.6), f satisfies (III).

Necessity. Let M satisfy the assumption of Theorem 3. The following proposition is known in [7] and [10].

PROPOSITION 5.1. *Let (M, g, V) be a quaternion Kaehler manifold. Then there exists a PR^3-bundle $\tilde{M}$ over M which is canonically associated to M. Moreover if the scalar curvature s of M is positive, $\tilde{M}$ has a Sasakian* 3*-structure.*

From the above proposition and Lemma 2.1, we get

PROPOSITION 5.2. *If M admits a non-trivial solution f for* (III), *then there exists a PR^3-bundle $\tilde{M}$ over M which is canonically associated to M and admits a Sasakian* 3*-structure.*

We denete by π the projection $\pi:\tilde{M}\rightarrow M$. We consider a solution f of the differential equation (III). Then the lift $\tilde{f}$ of f with respect to π satisfies (3.14). From (2.10) and (3.6), we have

(5.1) $\nabla_a f_c h_b{}^c{}_\alpha + \nabla_b f_c h_a{}^c{}_\alpha = 0$,

(5.1) $\nabla_c \nabla_b f_a - h_b{}^e{}_\alpha f_e h_{ca}{}^\alpha - h_a{}^e{}_\alpha f_e h_{cb}{}^\alpha + 2 f_c g_{ba} + f_b g_{ca} + f_a g_{cb} = 0$.

Thus, from (5.1) and (5.2), we get $L_{kji}=0$ (defined in § 3.). This implies that the function $\hat{f}$ is a non-trivial solution of the differential equation (II). Then $\tilde{M}$ is isometric to a space of constant curvature 1 by Theorem 2. If we take a universal covering M^* of $\tilde{M}$, then M^* is isometric to $S^{4m+3}(1)$. And the natural Sasakian 3-structure can be induced on M^* from that defined in M. Then from Proposition 4.1, M is isometric to $P^m(H)$.

Thus Theorem 3 is completely proved.

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