A CHARACTERIZATION OF $S^{n-1} \times S^1$ BY DIFFERENTIAL EQUATIONS

D. E. BLAIR AND M. OKUMURA

1. *Introduction*

It is well known (see, e.g., [3]) that if a complete Riemannian manifold admits a conformal non-Killing gradient vector field, then it is isometric to a Euclidean sphere. In general a conformal non-Killing vector field V satisfies $\mathscr{L}_V g = -2\lambda g$ for some function λ, where $\mathscr{L}$ denotes Lie differentiation and g the Riemannian metric. In the case that $V = \operatorname{grad}\lambda$, the formula $(\mathscr{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X)$ gives

$$\nabla_X V = -\lambda X,$$

where ∇ denotes covariant differentiation with respect to the Riemannian connexion of g.

The purpose of this note is to characterize the product $S^{n-1} \times S^1$ by similar differential equations.

THEOREM. *Let M^n, $n \geqslant 3$, be a compact Riemannian manifold with a regular unit Killing vector field Z. Then M^n admits a non-trivial function λ such that $V = \operatorname{grad}\lambda$ is orthogonal to Z and satisfies*

$$\nabla_X V = -\lambda X + \lambda g(Z, X) Z$$

if and only if M^n is globally isometric to $S^{n-1} \times S^1$.

The idea of the proof is to fibrate M^n as a circle bundle over a manifold M^{n-1} which is then shown to be a sphere. However, as the only principal circle bundle over a sphere of dimension $\geqslant 3$ is the trivial one, M^n, for $n \geqslant 4$, is then the product $S^{n-1} \times S^1$. (For example, in [2] Kobayashi shows that the set of all principal circle bundles over a simply connected manifold M form a group isomorphic to $H^2(M, Z)$.) For $n = 3$ the argument is slightly different.

2. *Proof of the Theorem* ($n \geqslant 4$)

Since the vector field Z is regular (i.e. each point of M^n has a neighbourhood such that the integral curve through the point passes through the neighbourhood only once) we obtain a foliation $\pi : M^n \to M^{n-1}$ of M^n over an $(n-1)$-dimensional manifold M^{n-1}. Now as M^n is compact and Z regular, its integral curves are circles; but Z being a Killing vector field of constant length gives

$$g(\nabla_Z Z, X) = -g(\nabla_X Z, Z) = 0$$

and hence that the integral curves of Z are also geodesics. Also since Z is Killing, the metric g is projectable to a metric g' on M^{n-1} and we have

$$g'(X, Y) \circ \pi = g(\tilde{\pi} X, \tilde{\pi} Y),$$

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$\tilde{\pi}$ denoting the horizontal lift with respect to the Riemannian connexion of g. Thus, by a theorem of R. Hermann [1], M^n is a principal circle bundle over M^{n-1}.

Now as $V = \operatorname{grad}\lambda$ is orthogonal to Z, we have $Z\lambda = 0$ and, by the differential equation, $\nabla_Z V = 0$. Moreover,

$$g(\nabla_V Z, X) = -g(\nabla_X Z, V) = g(Z, \nabla_X V)$$
$$= g(Z, -\lambda X + \lambda g(Z, X) Z) = 0$$

and hence

$$[Z, V] = \nabla_Z V - \nabla_V Z = 0.$$

Thus λ and V are projectable and we can define λ' and V' by $\lambda' \circ \pi = \lambda$ and $V' = \pi_* V$ (note that $V = \tilde{\pi} V'$).

Letting ∇' denote covariant differentiation with respect to the Riemannian connexion of g', the differential equations of the submersion are

$$\nabla_{\tilde{\pi}X}\tilde{\pi}Y = \tilde{\pi}\nabla'_X Y + A_{\tilde{\pi}X}\tilde{\pi}Y$$
$$\nabla_{\tilde{\pi}X} Z = A_{\tilde{\pi}X} Z + \operatorname{vert}(\nabla_{\tilde{\pi}X} Z)$$
$$\nabla_Z \tilde{\pi}X = A_{\tilde{\pi}X} Z + T_Z \tilde{\pi}X$$
$$\nabla_Z Z = T_Z Z + \operatorname{vert}(\nabla_Z Z)$$

where the decompositions are into horizontal and vertical parts and the tensors T and A are given in [4]. Now as Z is Killing and of constant length,

$$g(\nabla_Z \tilde{\pi}X, Z) = -g(\tilde{\pi}X, \nabla_Z Z) = g(\nabla_{\tilde{\pi}X} Z, Z) = 0;$$

thus the differential equations of the submersion become

$$\nabla_{\tilde{\pi}X}\tilde{\pi}Y = \tilde{\pi}\nabla'_X Y + A_{\tilde{\pi}X}\tilde{\pi}Y$$
$$\nabla_{\tilde{\pi}X} Z = A_{\tilde{\pi}X} Z$$
$$\nabla_Z \tilde{\pi}X = A_{\tilde{\pi}X} Z$$
$$\nabla_Z Z = 0.$$

We now differentiate V' and λ' on M^{n-1}.

$$\tilde{\pi}\nabla'_X V' = \nabla_{\tilde{\pi}X} V - A_{\tilde{\pi}X} V$$
$$= -\lambda\tilde{\pi}X - A_{\tilde{\pi}X} V$$
$$= \tilde{\pi}(-\lambda' X) - A_{\tilde{\pi}X} V$$

from which

$$\nabla'_X V' = -\lambda' X$$

and

$$A_{\tilde{\pi}X} V = 0.$$

We also have

$$(X\lambda')\circ\pi = (\tilde{\pi}X)\lambda = g(V, \tilde{\pi}X) = g'(V', X)\circ\pi$$

and hence

$$V' = \operatorname{grad}\lambda'.$$

Thus the base space M^{n-1} is isometric to a sphere, and for $n \geqslant 4$, the bundle $\pi : M^n \to M^{n-1}$ is trivial giving M^n as a product $S^{n-1}\times S^1$.

Conversely, it is easy to see that $S^{n-1}\times S^1$ admits a non-trivial solution of the given equations. Let Z be tangent to S^1 and z a 1-form such that $z(Z)=1$. Let g' be the usual metric on S^{n-1}; then $g=g'+z\otimes z$ is a metric on $S^{n-1}\times S^1$. Suppose V' and λ' satisfy $\nabla'_X V'=-\lambda' X$ and $V'=\operatorname{grad}\lambda'$ on S^{n-1}, then extend V' and λ' to $S^{n-1}\times S^1$ being zero on the second factor. Now computing $\nabla_X V$ and $X\lambda$ for $X=Z$ and X orthogonal to Z we see that the given equations are satisfied.

3. *The case* $n=3$

We first show that Z is parallel on M^3. As V' is a conformal non-Killing gradient vector field on S^2 (the projection of a constant vector field on R^3), we can find a (Killing) vector field W' orthogonal to V' except at their common isolated zeros. By §2 we have $A_{\tilde{\pi}W'}V=0$. Now as $A_X Y$ is skew-symmetric on horizontal vector fields, we have, extending by continuity, that $\nabla_{\tilde{\pi}X}\tilde{\pi}Y=\tilde{\pi}\nabla'_X Y$, i.e., A vanishes for horizontal vectors over S^2. Now

$$g(A_{\tilde{\pi}X}Z,\ \tilde{\pi}Y)=g(\nabla_{\tilde{\pi}X}Z,\ \tilde{\pi}Y)=-g(Z,\nabla_{\tilde{\pi}X}\tilde{\pi}Y)=0.$$

Also

$$g(A_{\tilde{\pi}X}Z,Z)=0$$

since $A_{\tilde{\pi}X}Z$ is horizontal. Thus $A=0$ on M^3 and hence Z is parallel on M^3.

As before, the set of all circle bundles over S^2 is isomorphic to $H^2(S^2,Z)\approx Z$. Moreover, all such bundles are known explicitly. They are the trivial one $S^2\times S^1$, the Hopf fibration $\pi: S^3\to S^2$ or they are obtained from the Hopf fibration as follows. Let G_p be the cyclic subgroup of S^1 of order p. Then S^3/G_p is a principal bundle over S^2 with group $S^1/G_p\approx S^1$ (*cf.* §5 of [2]).

Clearly M^3 is not S^3, as Z is a non-vanishing parallel vector field. If now M^3 were S^3/G_p for some p, then again as Z is parallel and non-vanishing, the simply connected covering space of M^3 would admit a non-vanishing parallel vector field and would therefore be non-compact. But the simply connected covering space of S^3/G_p is S^3. Thus M^3 must again be the trivial bundle over S^2, completing the proof.

References

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Michigan State University,
East Lansing, Michigan 48823.