A CHARACTERIZATION OF $S^{n-1} \times S^1$ BY DIFFERENTIAL EQUATIONS

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1. Introduction

It is well known (see, e.g., [3]) that if a complete Riemannian manifold admits a conformal non-Killing gradient vector field, then it is isometric to a Euclidean sphere. In general a conformal non-Killing vector field $V$ satisfies $\mathcal{L}_V g = -2\lambda g$ for some function $\lambda$, where $\mathcal{L}$ denotes Lie differentiation and $g$ the Riemannian metric. In the case that $V = \text{grad} \lambda$, the formula $(\mathcal{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X)$ gives

$$\nabla_X V = -\lambda X,$$

where $\nabla$ denotes covariant differentiation with respect to the Riemannian connexion of $g$.

The purpose of this note is to characterize the product $S^{n-1} \times S^1$ by similar differential equations.

**Theorem.** Let $M^n$, $n \geq 3$, be a compact Riemannian manifold with a regular unit Killing vector field $Z$. Then $M^n$ admits a non-trivial function $\lambda$ such that $V = \text{grad} \lambda$ is orthogonal to $Z$ and satisfies

$$\nabla_X V = -\lambda X + \lambda g(Z, X) Z$$

if and only if $M^n$ is globally isometric to $S^{n-1} \times S^1$.

The idea of the proof is to fibrate $M^n$ as a circle bundle over a manifold $M^{n-1}$ which is then shown to be a sphere. However, as the only principal circle bundle over a sphere of dimension $\geq 3$ is the trivial one, $M^n$, for $n \geq 4$, is then the product $S^{n-1} \times S^1$. (For example, in [2] Kobayashi shows that the set of all principal circle bundles over a simply connected manifold $M$ form a group isomorphic to $H^2(M, Z)$.) For $n = 3$ the argument is slightly different.

2. Proof of the Theorem ($n \geq 4$)

Since the vector field $Z$ is regular (i.e. each point of $M^n$ has a neighbourhood such that the integral curve through the point passes through the neighbourhood only once) we obtain a foliation $\pi: M^n \rightarrow M^{n-1}$ of $M^n$ over an $(n-1)$-dimensional manifold $M^{n-1}$. Now as $M^n$ is compact and $Z$ regular, its integral curves are circles; but $Z$ being a Killing vector field of constant length gives

$$g(\nabla_Z Z, X) = -g(\nabla_X Z, Z) = 0$$

and hence that the integral curves of $Z$ are also geodesics. Also since $Z$ is Killing, the metric $g$ is projectable to a metric $g'$ on $M^{n-1}$ and we have

$$g'(X, Y) \circ \pi = g(\tilde{\pi}X, \tilde{\pi}Y),$$

Received 4 September, 1972.

\( \tilde{\pi} \) denoting the horizontal lift with respect to the Riemannian connexion of \( g \). Thus, by a theorem of R. Hermann [1], \( M^n \) is a principal circle bundle over \( M^{n-1} \).

Now as \( V = \text{grad} \lambda \) is orthogonal to \( Z \), we have \( Z\lambda = 0 \) and, by the differential equation, \( \nabla_Z V = 0 \). Moreover,

\[
g(\nabla_Y Z, X) = -g(\nabla_X Z, Y) = g(Z, \nabla_X Y)
\]

and hence

\[
[Z, V] = \nabla_Z V - \nabla_V Z = 0.
\]

Thus \( \lambda \) and \( V \) are projectable and we can define \( \lambda' \) and \( V' \) by \( \lambda' \circ \pi = \lambda \) and \( V' = \pi_* V \) (note that \( V = \tilde{\pi}V' \)).

Letting \( V' \) denote covariant differentiation with respect to the Riemannian connexion of \( g' \), the differential equations of the submersion are

\[
\nabla_{\tilde{\pi}X} \tilde{\pi}Y = \tilde{\pi}\nabla_Y X + A_{\tilde{\pi}X} \tilde{\pi}Y
\]

\[
\nabla_{\tilde{\pi}X} Z = A_{\tilde{\pi}X} Z + \text{vert} (\nabla_{\tilde{\pi}X} Z)
\]

\[
\nabla_Z \tilde{\pi}X = A_{\tilde{\pi}X} Z + T_Z \tilde{\pi}X
\]

\[
\nabla_Z Z = T_Z Z + \text{vert} (\nabla_Z Z)
\]

where the decompositions are into horizontal and vertical parts and the tensors \( T \) and \( A \) are given in [4]. Now as \( Z \) is Killing and of constant length,

\[
g(\nabla_Z \tilde{\pi}X, Z) = -g(\tilde{\pi}X, \nabla_Z Z) = g(\nabla_{\tilde{\pi}X} Z, Z) = 0;
\]

thus the differential equations of the submersion become

\[
\nabla_{\tilde{\pi}X} \tilde{\pi}Y = \tilde{\pi}\nabla_Y X + A_{\tilde{\pi}X} \tilde{\pi}Y
\]

\[
\nabla_{\tilde{\pi}X} Z = A_{\tilde{\pi}X} Z
\]

\[
\nabla_Z \tilde{\pi}X = A_{\tilde{\pi}X} Z
\]

\[
\nabla_Z Z = 0.
\]

We now differentiate \( V' \) and \( \lambda' \) on \( M^{n-1} \).

\[
\tilde{\pi}\nabla_X V' = \nabla_{\tilde{\pi}X} V - A_{\tilde{\pi}X} V
\]

\[
= -\lambda \tilde{\pi}X - A_{\tilde{\pi}X} V
\]

\[
= \tilde{\pi}(-\lambda' X) - A_{\tilde{\pi}X} V
\]

from which

\[
\nabla_X V' = -\lambda' X
\]

and

\[
A_{\tilde{\pi}X} V = 0.
\]

We also have

\[ (\tilde{\pi}X') \circ \pi = (\tilde{\pi}X) \lambda = g(V, \tilde{\pi}X) = g'(V', X) \circ \pi \]

and hence

\[ V' = \text{grad} \lambda'. \]

Thus the base space \( M^{n-1} \) is isometric to a sphere, and for \( n \geq 4 \), the bundle \( \pi: M^n \to M^{n-1} \) is trivial giving \( M^n \) as a product \( S^{n-1} \times S^1 \).
Conversely, it is easy to see that $S^{n-1} \times S^1$ admits a non-trivial solution of the given equations. Let $Z$ be tangent to $S^1$ and $z$ a 1-form such that $z(Z) = 1$. Let $g'$ be the usual metric on $S^{n-1}$; then $g = g' + z \otimes z$ is a metric on $S^{n-1} \times S^1$. Suppose $V'$ and $\lambda'$ satisfy $\nabla_x V' = -\lambda'X$ and $V' = \text{grad} \lambda'$ on $S^{n-1}$, then extend $V'$ and $\lambda'$ to $S^{n-1} \times S^1$ being zero on the second factor. Now computing $\nabla_x V$ and $X\lambda$ for $X = Z$ and $X$ orthogonal to $Z$ we see that the given equations are satisfied.

3. The case $n = 3$

We first show that $Z$ is parallel on $M^3$. As $V'$ is a conformal non-Killing gradient vector field on $S^2$ (the projection of a constant vector field on $R^3$), we can find a (Killing) vector field $W'$ orthogonal to $V'$ except at their common isolated zeros. By §2 we have $A_{\tilde{w}'} V = 0$. Now as $A_X Y$ is skew-symmetric on horizontal vector fields, we have, extending by continuity, that $\nabla_{\tilde{w}'} \pi Y = \pi \nabla_{\tilde{w}'} Y$, i.e., $A$ vanishes for horizontal vectors over $S^2$. Now

$$g(A_{\tilde{w}'} Z, \pi Y) = g(\nabla_{\tilde{w}'} Z, \pi Y) = -g(Z, \nabla_{\tilde{w}'} \pi Y) = 0.$$  

Also

$$g(A_{\tilde{w}'} Z, Z) = 0$$

since $A_{\tilde{w}'} Z$ is horizontal. Thus $A = 0$ on $M^3$ and hence $Z$ is parallel on $M^3$.

As before, the set of all circle bundles over $S^2$ is isomorphic to $H^2(S^2, Z) \approx Z$. Moreover, all such bundles are known explicitly. They are the trivial one $S^2 \times S^1$, the Hopf fibration $\pi: S^3 \to S^2$ or they are obtained from the Hopf fibration as follows. Let $G_p$ be the cyclic subgroup of $S^1$ of order $p$. Then $S^3/G_p$ is a principal bundle over $S^2$ with group $S^1/G_p \approx S^1$ (cf. §5 of [2]).

Clearly $M^3$ is not $S^3$, as $Z$ is a non-vanishing parallel vector field. If now $M^3$ were $S^3/G_p$ for some $p$, then again as $Z$ is parallel and non-vanishing, the simply connected covering space of $M^3$ would admit a non-vanishing parallel vector field and would therefore be non-compact. But the simply connected covering space of $S^3/G_p$ is $S^3$. Thus $M^3$ must again be the trivial bundle over $S^2$, completing the proof.

References


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