M-690 FINAL EXAMINATION

II-Semester, 1427-1428

Department of Mathematics (Women's Section)

College of Science, King Saud University

Time 3 Hours **Max. Marks-50**

Note: *Attempt all questions*

Q.1. Show that on an orientable n-dimensional smooth manifold M there exists a $\Omega \in \Lambda^n(M)$ such that at each point $p \in M$, $\Omega_p \neq 0$. Use this to show that the unit sphere S^n is an oreintable manifold.

Q.2. Let M, $M^{'}$ be two orientable paracompact Reimannian manifolds and $f : M \rightarrow M^{'}$ be an orientation preserving diffeomorphism. Then show that

$$\int_M f^*\Omega = \int_{M'} \Omega$$

where $\Omega \in \Lambda^n(M^{'})$ has compact support. Also show that for a compact orientable M without boundary and a smooth vector field $X \in \mathfrak{X}(M)$ and a volume form ω on M

$$\int_M \pounds_X \omega = 0$$

Q.3. Explain what do you mean by a smooth manifold with boundary. If $U \subset M$ is a domain with smooth boundary ∂U, show that ∂U is a smooth manifold of dimension equal to $\dim M - 1$, and that if M is paracompact and orientable then ∂U is also orientable.

Q.4. Explain what do you mean by a variation of a submanifold M with boundary ∂M of a Riemannian manifold and how it gives rise to volume function of the variation. Obtain first variation formula for a compact orientable submanifold of a Riemannian manifold and deduce that minimal submanifolds are the critical points of volume function.

Q.5. What do you mean by a stable minimal submanifold of a Riemannian manifold? Show that the totally geodesic sphere S^n in S^{n+k} is not stable. Compute the index and nullity of S^n.

Q.6. What is a Laplacian operator Δ on a Riemannian manifold (M, g)? If M is orientable and compact, show that Δ is self adjoint with respect to the inner product $\langle , \rangle$ defined on the space of smooth functions by

$$\langle f, g \rangle = \int_M fg dV$$

Give an example of a Riemannian manifold and an eigen function of the Laplacian operator Δ on this Riemannian manifold..