M-673 FINAL EXAMINATION

I-Semester, 1427-1428

Department of Mathematics (Women's Section)

College of Science, King Saud University

Time 3 Hours **Max. Marks-50**

Note: *Attempt all questions*

Q.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Show that $\mathfrak{g}$ as a vector space is isomorphic to the tangent space T_eG at the identity $e \in G$. Also give an example of a smooth vector field on G that is not left invariant to support in general $\mathfrak{g} \subset \mathfrak{X}(G)$ is a proper subset.

Q.2. Explain what do you mean by a one-parameter subgroup of a Lie group. Suppose $O(G)$ be the set of all one-parameter subgroups of a Lie group G. Show that there is a one-to-one correspondence between $O(G)$ and T_eG, and use this one-to-one correspondence to define the exponential mappong $\exp : \mathfrak{g} \to G$.

Q.3. (a) Let $\phi : G \to Aut(V)$ be a representation of a Lie group G. Show that

$$d\phi(X) = \lim_{t \to 0} \frac{\phi(\exp tX) - I}{t}$$

for $X \in \mathfrak{g}$.

(b) For a Lie group G with Lie algebra $\mathfrak{g}$ verify that $ad(X)(Y) = [X, Y]$, $X, Y \in \mathfrak{g}$.

Q.4. (a) For a connected Lie group G, show that its center $Z(G)$ is a Lie group with Lie algebra $z(\mathfrak{g})$ the center of the Lie algebra $\mathfrak{g}$.

(b) Explain what do you mean by a semisimple Lie group and show that the Lie group $SU(2)$ is semisimple. Also give an example of a Lie group which is not semisimple(without proof)

Q.5. For a bi-invariant metric on a Lie group G show that
(i) the covariant derivative with respect the connection corresponding to this metric is given by

$$\nabla_X Y = \frac{1}{2}[X, Y], \qquad X, Y \in \mathfrak{g}$$

(ii) Each one parameter subgroup of G is a geodesic.

(iii) For a plain section π spanned by orthonormal set $\{X, Y\} \subset \mathfrak{g}$, the sectional curvature is given by

$$K(\pi) = \frac{1}{4} \left\| [X, Y] \right\|^2$$

Q.6. Let G be a special type of Lie group (that is Heisenberg type of Lie group). Assume the existence of the linear map $l : \mathfrak{g} \to R$ that satisfies $[X, Y] = l(X)Y - l(Y)X$, $X, Y \in \mathfrak{g}$, and that G is equipped with a left invariant metric $\langle , \rangle$. Then show that the Riemannian curvature tensor with respect to this metric satisfies

$$R(X, Y)Z = - \left\| l \right\|^2 \{ < Y, Z > X - < X, Z > Y \}$$

for $X, Y \in \ker l = \mathcal{I}$.