Q.1. (a) Show that $M = \{ x = (x_1, ..., x_n) \in \mathbb{R}^n - \{0\} : \|x\|^2 = 2x_1^2 \}$ is a smooth manifold.

(b) Explain what do you mean by an immersion, a submersion and give examples of each and an example of a submersion which is not an immersion.

Q.2. (a) Let $M$ and $N$ be two smooth manifolds and $f : M \to N$ be a continuous function. For $p \in M$, let $(U, \varphi)$ and $(V, \psi)$ be charts around $p$ and $f(p)$ with $f(U) \subset V$ and $y^1, ..., y^n$ be local coordinates on $V$. If $f^i = y^i \circ f$ are smooth functions at $p$, $i = 1, ..., n$, then show that $f$ is smooth at $p$.

(b) Let $f : \mathbb{R}^2 - \{0\} \to S^1$ be

$$f(x, y) = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)$$

and $p = (1, 1)$. Find $df_p \left( \left( \frac{\partial}{\partial x} \right)_p \right)$.

Q.3. (a) Let $M$ be a connected manifold and $f : M \to \mathbb{R}$ be a non-constant smooth function. Show that there is a point $p \in M$ and a chart $(U, \varphi)$ around $p$ with local coordinates $x^1, ..., x^n$ with $x^1 = f$.

(b) For the smooth function $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x, y, z) = (x^2 + y^2, xz, yz)$$

and the point $p = (1, 1, 1)$ show that there exists a neighbourhood $U$ of $p$ such that $f : U \to \mathbb{R}^3$ is one to one.