Q.1. (a) Show that the set
\[ O(n) = \{ A \in GL(n, R) : AA^T = I \} \]
is a smooth manifold. What is the dimension of \( O(n) \)?

(b) Let \( \alpha : \left( -\frac{\pi}{4}, \frac{\pi}{4} \right) \to S^2 \) be \( \alpha(t) = \left( \frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \sin t \right) \) and \( \pi : S^2 \to RP^2 \) be the projection. For the smooth curve \( \beta(t) = \pi(\alpha(t)) \) on \( RP^2 \) and a chart \((U, \varphi)\) around \( \beta(0) \) with local coordinates \( x^1, x^2 \), if
\[ \dot{\beta}(0) = \lambda^1 \left( \frac{\partial}{\partial x^1} \right)_{\beta(0)} + \lambda^2 \left( \frac{\partial}{\partial x^2} \right)_{\beta(0)} \]
find the constants \( \lambda^i \).

Q.2. (a) Show that if \( f : M \to N \) is a diffeomorphism then \( df_p : T_pM \to T_{f(p)}N \) is an isomorphism for each \( p \in M \). Is the converse true?

(b) Let \((U, \varphi)\) and \((V, \psi)\) be two charts and \( p \in U \cap V \). Find the matrix for \( d(\varphi \circ \psi^{-1})_{\psi(p)} \).

Q.3. (a) Show that the tangent bundle \( TM \) of a smooth manifold \( M \) is a smooth manifold (do not verify the Hausdorff condition).

(b) Use the projection map \( \pi : TM \to M \) to show that the fiber \( \pi^{-1}(\{p\}) = T_pM, p \in M \) is a smooth manifold (do not use the isomorphism of \( T_pM \) to \( R^n \)).
Q.4 (a) If $X : C^\infty(M) \to C^\infty(M)$ is a derivation of the ring $C^\infty(M)$ show that $X$ is a smooth vector field.

(b) Let $(U, \varphi)$ be a chart on $RP^2$ with $U = \{[(x, y, z) : z \neq 0]\}$ and $X \in \mathfrak{X}(RP^2)$ be

$$X = \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$$

where $x^1, x^2$ are local coordinates on $U$. Find the integral curve of $X$ passing through the point $[(0, 1, 1)]$.

Q.5 (a) Let $f : M \to N$ be a smooth map. Explain why in general $df(X)$ does not define a vector field on $N$ for $X \in \mathfrak{X}(M)$? Also explain what do you mean by $f$-related vector fields. Show that if $X, Y \in \mathfrak{X}(M)$ are $f$-related to $Z, W \in \mathfrak{X}(N)$ respectively, then $[X, Y]$ is $f$-related to $[Z, W]$.

(b) If $\{\varphi_t\}$ is a one-parameter group of transformations on $M$, then show that $\sigma(t) = \varphi_t(p), p \in M$ is an integral curve of the vector field $X$ induced by $\{\varphi_t\}$.

Q.6 (a) Show that $\{\varphi_t\}$,

$$\varphi_t(x, y) = (xe^{2yt}, y)$$

is a one-parameter group of transformations on $R^2$ and find the vector field induced by $\{\varphi_t\}$.

(b) Let $M$ be a smooth manifold and $p \in M$. For a tangent vector $u \in T_pM$ show that there is a vector field $X \in \mathfrak{X}(M)$ such that

$$u = X(p)$$