# M-573 FINAL EXAMINATION, II-SEMESTER (1430)

Department of Mathematics, College of Science (Women's Section)

King Saud University

Time: 3 hours | Max.Marks-50

Note: *Attempt all questions*

**Q.1.** **(a)** Show that the set

$$O(n) = \left\{ A \in GL(n, R) : AA^T = I \right\}$$

is a smooth manifold. What is the dimension of $O(n)$?

**(b)** Let $\alpha : \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \to S^2$ be $\alpha(t) = \left(\frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin t\right)$ and $\pi : S^2 \to RP^2$ be the projection. For the smooth curve $\beta(t) = \pi(\alpha(t))$ on $RP^2$ and a chart $(U, \varphi)$ around $\beta(0)$ with local coordinates $x^1, x^2$, if

$$\dot{\beta}(0) = \lambda^1 \left(\frac{\partial}{\partial x^1}\right)_{\beta(0)} + \lambda^2 \left(\frac{\partial}{\partial x^2}\right)_{\beta(0)}$$

find the constants $\lambda^i$.

**Q.2.** **(a)** Show that if $f : M \to N$ is a diffeomorphism then $df_p : T_pM \to T_{f(p)}N$ is an isomorphism for each $p \in M$. Is the converse true?

**(b)** Let $(U, \varphi)$ and $(V, \psi)$ be two charts and $p \in U \cap V$. Find the matrix for $d(\varphi \circ \psi^{-1})_{\psi(p)}$.

**Q.3.** **(a)** Show that the tangent bundle $TM$ of a smooth manifold $M$ is a smooth manifold (do not verify the Hausdorff condition).

**(b)** Use the projection map $\pi : TM \to M$ to show that the fiber $\pi^{-1}(\{p\}) = T_pM$, $p \in M$ is a smooth manifold (do not use the isomorphism of $T_pM$ to $R^n$).

**Q.4 (a)** If $X : C^\infty(M) \to C^\infty(M)$ is a derivation of the ring $C^\infty(M)$ show that $X$ is a smooth vector field.

**(b)** Let $(U, \varphi)$ be a chart on $RP^2$ with $U = \{[(x, y, z)] : z \neq 0\}$ and $X \in \mathfrak{X}(RP^2)$ be

$$X = \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$$

where $x^1, x^2$ are local coordinates on $U$. Find the integral curve of $X$ passing through the point $[(0, 1, 1)]$.

**Q.5 (a)** Let $f : M \to N$ be a smooth map. Explain why in general $df(X)$ does not define a vector field on $N$ for $X \in \mathfrak{X}(M)$? Also explain what do you mean by $f$-related vector fields. Show that if $X, Y \in \mathfrak{X}(M)$ are $f$-related to $Z, W \in \mathfrak{X}(N)$ respectively, then $[X, Y]$ is $f$-related to $[Z, W]$.

**(b)** If $\{\varphi_t\}$ is a one-parameter group of transformations on $M$, then show that $\sigma(t) = \varphi_t(p)$, $p \in M$ is an integral curve of the vector field $X$ induced by $\{\varphi_t\}$.

**Q.6 (a)** Show that $\{\varphi_t\}$,

$$\varphi_t(x, y) = \left(xe^{2yt}, y\right)$$

is a one-parameter group of transformations on $R^2$ and find the vector field induced by $\{\varphi_t\}$.

**(b)** Let $M$ be a smooth manifold and $p \in M$. For a tangent vector $u \in T_pM$ show that there is a vector field $X \in \mathfrak{X}(M)$ such that

$$u = X(p)$$