M-570 FINAL EXAMINATION

I-Semester, 1430

Department of Mathematics

College of Science, King Saud University

Time 3 Hours **Max. Marks-50**

Note: *Attempt all questions*

Q.1.(a) Let M be an n-dimensional smooth manifold and $p \in M$. For a smooth function $f \in C^{\infty}(p)$ and a chart (U, φ) with local coordinates $x^1, .., x^n$, show that there exist smooth functions $g_1, , g_n \in C^{\infty}(\varphi(p))$ satisfying

$$f = f(p) + \sum_{i=1}^{n} \left(x^i - x^i(p)\right) g_i \circ \varphi$$

(b) For charts (U, φ) and (V, ψ), $U \cap V \neq \phi$ on a smooth manifold M with local coordinates $x^1, , x^n$ and $y^1, , y^n$ repectively, show that the matrix

$$A = \begin{bmatrix} \frac{\partial y^1}{\partial x^1}(p) & .. & .. & \frac{\partial y^1}{\partial x^n}(p) \\ .. & .. & .. & . \\ .. & .. & .. & . \\ \frac{\partial y^n}{\partial x^1}(p) & & & \frac{\partial y^n}{\partial x^n}(p) \end{bmatrix}$$

is invertible.

Q.2. (a) Let $f : S^2 \to R$ be $f(x, y, z) = x + y - z$ and $p = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \in S^2$. For a chart (U, φ) around p with local coordinates x^1, x^2 and $X_p = \left(\frac{\partial}{\partial x^1}\right)_p - \left(\frac{\partial}{\partial x^2}\right)_p$ find $X_p(f)$.

(b) Find df_p and ascertain whether the point p is a regular point or a critical point.

Q.3. (a) Let $h : S^1 \to S^2$ be $h(x, y) = \left(\frac{x-y}{\sqrt{3}}, \frac{x+y}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $q = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \in S^1$. Find $h^*(f)(q)$ for the function f in Q.2(a).

(b) Let $U = S^1 - \{(0, 1)\}$, $\varphi : U \to R$ and t be the local cordinate on the chart (U, φ). Find $\frac{\partial}{\partial t}(h^*(f))(q)$.

(c) Find the matrix for dh_q.

Q.4. (a) Let $f, g : R^2 \to R$ be $f(x, y) = \frac{1}{2}(x^2 - y^2)$ and $g(x, y) = xy$. For a point $p \in R^2 - \{(0, 0)\}$, show that there exists a chart (U, φ) around p with local coordinates f, g.

(b) Let $\varphi : R^2 \to R^2$, be $\varphi(x, y, z, w) = (x + y - z, y + z - w)$. Show that φ is a submersion. Is $M = \varphi^{-1}\{(0, 0)\}$ a smooth manifold?

Q.5.(a) Let $f : M \to N$ be an immersion. Then show that for any $p \in M$, there is a neighbourhood U of p, such that $f : U \to N$ is one-to-one.

(b) If $f : M \to N$ is a smooth map and $q \in N$ is a regular value of f, then show that $L = f^{-1}\{q\}$ is a smooth manifold.

Q.6.(a) Let M be a smooth manifold and $X : C^{\infty}(M) \to C^{\infty}(M)$ be a derivation of the ring $C^{\infty}(M)$. Show that X is a smooth vector field.

(b) Let $U = \left\{[(x, y, x)] \in RP^2 : z \neq 0\right\}$ and $\varphi : U \to R^2$ be $\varphi\left([(x, y, z)]\right) = \left(\frac{x}{z}, \frac{y}{z}\right)$ with local coordinates x^1, x^2. If $X \in \mathfrak{X}(RP^2)$ has local expression

$$X = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$$

find an integral curve of X passing through the point $[(-1, 1, 1)]$.