TANGENT BUNDLE OF A HYPERSURFACE OF A EUCLIDEAN SPACE

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Abstract: We use the fact that the tangent bundle TM of an orientable hypersurface M in the Euclidean space R^{n+1} is a submanifold of the Euclidean space R^{2n+2}, and use the induced metric on TM as submanifold to study its geometry. This induced metric is not a natural metric in the sense that the projection $\pi : TM \to M$ is not a Riemannian submersion (which holds for Sasaki and other metrics, used to study geometry of the tangent bundle). First we prove that there is a reduction in the codimension of the submanifold TM and thus the tangent bundle TM is a hypersurface of the Euclidean space R^{2n+1}. As a consequence of our study, we infer that the induced metric on TS^n the tangent bundle of the unit sphere S^n makes TS^n a Riemannian manifold of nonnegative sectional curvature. We also obtain a condition under which the tangent bundle TM of a hypersurface M in a Euclidean space is trivial.

1. INTRODUCTION

The study of geometry of the tangent bundle of a Riemannian manifold started with the work of Sasaki [9], and since then the tangent bundle has become focus of study with this metric. Specially after the work of Dombrowoski [3], who has introduced a nice theory of linking the geometry of the tangent bundle with Sasaki metric to the geometry of the base manifold, many mathematicians have studied the geometry of the tangent bundle with various aspects (cf. the survey article [5] and references therein). Since there is a naturally associated almost complex structure J to the tangent bundle TM of a Riemannian manifold M, one naturally expects fairly good properties associated to this almost complex structure vis-a-vis the complex geometry. However, the Sasaki metric on TM offers a significant obstruction on the almost complex structure and does not allow it even to be a complex structure unless the base manifold is flat. This deficiency in the Sasaki metric lead mathematicians to search for other metrics on the tangent bundle such as Cheeger-Gromoll metric, Oproiu metric (cf. [1], [5], [8], [10]). All these known metrics on TM are compatible with the smooth projection $\pi : TM \to M$ in the sense that the projection becomes a Riemannian submersion and therefore these metrics are

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called the natural metrics. Recently in [2], efforts are made to study the geometry of the tangent bundle of a hypersurface M in the Euclidean space R^{n+1}, where the authors have shown that the induced metric on its tangent bundle TM as submanifold of the Euclidean space R^{2n+2} is not a natural metric. The tangent bundle TM of a Riemannian manifold M, being noncompact it can not admit a Riemannian metric of strictly positive sectional curvature. Therefore a natural question would be to find a metric on the tangent bundle TM which has nonnegative sectional curvature. In this paper we extend the study initiated in [2] on the geometry of the tangent bundle TM of an immersed orientable hypersurface M in the Euclidean space R^{n+1} and first we prove a codimension reduction theorem, that the tangent bundle TM of the hypersurface M is the hypersurface of the Euclidean space R^{2n+1} and then subsequently we answer the question stated above for the tangent bundle TS^n of the unit sphere S^n in the Euclidean space R^{n+1} by showing that TS^n admits a Riemannian metric of non-negative sectional curvature (cf. Corollary 4.1).

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2. PRELIMINARIES

Let (M,g) be a Riemannian manifold and TM be its tangent bundle with projection map $\pi : TM \to M$. Then for each $(p,u) \in TM$, the tangent space $T_{(p,u)}TM = \mathfrak{H}_{(p,u)} \oplus \mathfrak{V}_{(p,u)}$, where $\mathfrak{V}_{(p,u)}$ is kernel of $d\pi_{(p,u)} : T_{(p,u)}TM \to T_pM$ and $\mathfrak{H}_{(p,u)}$ is the kernel of the connection map $K_{(p,u)} : T_{(p,u)}(TM) \to T_pM$ with respect to the Riemannian connection on (M,g). The subspaces $\mathfrak{H}_{(p,u)}$, $\mathfrak{V}_{(p,u)}$ are called the horizontal and vertical subspaces respectively. Consequently, the Lie algebra of smooth vector fields $\mathfrak{X}(TM)$ on the tangent bundle TM admits the decomposition $\mathfrak{X}(TM) = \mathfrak{H} \oplus \mathfrak{V}$, where $\mathfrak{H}$ is called the horizontal distribution and $\mathfrak{V}$ is called the vertical distribution on the tangent bundle TM. For each $X_p \in T_pM$, the horizontal lift of X_p to a point $z = (p,u) \in TM$ is the unique vector $X_z^h \in \mathfrak{H}_z$ such that $d\pi(X_z^h) = X_p \circ \pi$ and the vertical lift of X_p to a point $z = (p,u) \in TM$ is the unique vector $X_z^v \in \mathfrak{V}_z$ such that $X_z^v(df) = X_p(f)$ for all functions $f \in C^\infty(M)$, where df is the function defined by $(df)(p,u) = u(f)$. Also, for a vector field $X \in \mathfrak{X}(M)$, the horizontal lift of X is a vector field $X^h \in \mathfrak{X}(TM)$, whose value at a point (p,u) is the horizontal lift of $X(p)$ to (p,u), the vertical lift X^v of X is defined similarly. For $X \in \mathfrak{X}(M)$ the horizontal and vertical lifts X^h, X^v of X are the uniquely determined vector fields on TM satisfying

$$d\pi(X_z^h) = X_{\pi(z)},\ K(X_z^h) = 0_{\pi(z)},\ \ d\pi(X_z^v) = 0_{\pi(z)},\ K(X_z^v) = X_{\pi(z)}.$$

Also for a smooth function $f \in C^\infty(M)$ and vector fields $X, Y \in \mathfrak{X}(M)$, we have $(fX)^h = (f \circ \pi)X^h$, $(fX)^v = (f \circ \pi)X^v$, $(X+Y)^h = X^h + Y^h$ and

$(X+Y)^v = X^v + Y^v$. If $\dim M = m$ and (U, ϕ) is a chart on M with local coordinates $x^1, x^2, ..., x^m$, then $(\pi^{-1}(U), \bar{\Phi})$ is a chart on TM with local coordinates $x^1, ..., x^m, y^1, ..., y^m$, where $x^i = x^i \circ \pi$ and $y^i = dx^i$, $i = 1, ..., m$. Throughout this paper we use Einstein convention for summation, that is, the repeated indices are summed over their ranges. For horizontal and vertical lifts of smooth vector fields, we have the following:

Lemma 2.1 ([5]) Let (M, g) be a Riemannian manifold and $X, Z \in \mathfrak{X}(M)$, which locally be represented by $X = \xi^i \frac{\partial}{\partial x^i}$ and $Z = \eta^i \frac{\partial}{\partial x^i}$. Then the vertical and horizontal lifts X^v and X^h of X at the point $Z \in TM$ are given by

$$(X^v)_Z = \xi^i \frac{\partial}{\partial y^i}, \qquad (X^h)_Z = \xi^i \frac{\partial}{\partial x^i} - \xi^j \eta^k \Gamma^i_{jk} \frac{\partial}{\partial y^i},$$

where the coefficients Γ^i_{jk} are the Christoffel symbols of the Riemannian connection ∇ on (M, g).

A Riemannian metric $\bar{g}$ on the tangent bundle TM is said to be a natural metric with respect to g on M if $\bar{g}_{(p,u)}(X^h, Y^h) = g_p(X, Y)$ and $\bar{g}_{(p,u)}(X^h, Y^v) = 0$, for all vector fields $X, Y \in \mathfrak{X}(M)$ and $(p, u) \in TM$, that is the projection map $\pi : TM \to M$ is the Riemannian submersion [7].

Consider the Euclidean space $\big(R^{n+1}, \langle\ ,\ \rangle\big)$, where $\langle\ ,\ \rangle$ is the Euclidean metric and let M be an immersed hypersurface of $\big(R^{n+1}, \langle\ ,\ \rangle\big)$ with the immersion $f : M \to R^{n+1}$. Then we have the smooth maps

$$F = df : TM \to R^{2n+2}, \quad \widetilde{\pi} : R^{2n+2} \to R^{n+1},$$

defined by $F(p, X_p) = (f(p), df_p(X_p))$ and $\widetilde{\pi}(x, y) = x$ for $x, y \in R^{n+1}$, where $df_p : T_pM \to R$ is the differential of the map f at $p \in M$. Clearly $f \circ \pi = \widetilde{\pi} \circ F$ holds, where $\pi : TM \to M$ is the projection of the tangent bundle. It is easy to check that the projection $\widetilde{\pi} : \big(R^{2n+2}, \langle\ ,\ \rangle\big) \to \big(R^{n+1}, \langle\ ,\ \rangle\big)$ is a Riemannian submersion (cf. [7]).

If $x^1, ..., x^n$ are local coordinates on M, then the corresponding local coordinates on TM are $x^1, x^2, ..., x^n, y^1, y^2, ..., y^n$, where $x^i = x^i \circ \pi$, $y^i = dx^i$, $i = 1, ..., n$. Similarly if $u^1, ..., u^{n+1}$ are coordinates on R^{n+1} then we get a corresponding coordinates $u^1, ..., u^{n+1}, v^1, ..., v^{n+1}$ on R^{2n+2} which obey the following for the vertical and horizontal lifts of the coordinate vector fields

$$\begin{aligned} \left(\frac{\partial}{\partial u^i}\right)^v &= \frac{\partial}{\partial v^i} \\ \left(\frac{\partial}{\partial u^i}\right)^h &= \frac{\partial}{\partial u^i}, \qquad i = 1, ..., n+1. \end{aligned}$$

The following theorem is a consequence of the fact that an immersion of a smooth manifold M into the smooth manifold N gives an immersion of TM in TN.

Theorem 2.1 [2] Let M be an orientable hypersurface of the Euclidean space R^{n+1} with immersion $f: M \to R^{n+1}$. Then the map $F: TM \to R^{2n+2}$ defined by

$$F = \left(f^1 \circ \pi, f^2 \circ \pi, ..., f^{n+1} \circ \pi, (\frac{\partial f^1}{\partial x^i} \circ \pi) y^i, ..., (\frac{\partial f^{n+1}}{\partial x^i} \circ \pi) y^i) \right)$$

is an immersion with the matrix for the differential $dF_P : T_P(TM) \to T_{F(P)}(R^{2n+2})$ at $P = (p, X_p) \in TM$, is the $(2n+2) \times 2n$ matrix

$$dF_P = \begin{bmatrix} df_{p_{(n+1)\times n}} & 0_{(n+1)\times n} \\ (\frac{\partial^2 f^i}{\partial x^j \partial x^k}(p) y^k(P))_{(n+1)\times n} & df_{p_{(n+1)\times n}} \end{bmatrix}.$$

Thus the tangent bundle TM of an orientable hypersurface M of the Euclidean space R^{n+1} is a submanifold of R^{2n+2}. We denote the induced Riemannian metrics on M and TM by g and $\overline{g}$ respectively. Let us denote by $D, \bar{D}$ the Euclidean connections on R^{n+1}, R^{2n+2} respectively, by $\nabla, \bar{\nabla}$ the Riemannian connections on M, TM respectively and by N the unit normal vector field of the orientable hypersurface M. For the hypersurface M of the Euclidean space R^{n+1} we have the following Gauss and Weingarten formulae

$$D_X Y = \nabla_X Y + \langle S(X), Y \rangle N \tag{2.1}$$

$$D_X N = -S(X) \tag{2.2}$$

where $X, Y \in \mathfrak{X}(M)$ and S denotes the shape operator (Weingarten map). Similarly for the submanifold TM of the Euclidean space R^{2n+2} we have the Gauss and Weingarten formulae

$$\bar{D}_X Y = \bar{\nabla}_X Y + h(X, Y) \tag{2.3}$$

$$\bar{D}_X \hat{N} = -\bar{S}_{\hat{N}}(X) + \nabla^{\perp}_X \hat{N} \tag{2.4}$$

where $X, Y \in \mathfrak{X}(TM)$ and $\bar{S}_{\hat{N}}$ denotes the Weingarten map in the direction of the normal $\hat{N}$ which is related to the second fundamental form h by

$$\left\langle h(X, Y), \hat{N} \right\rangle = \overline{g}\left(\bar{S}_{\hat{N}}(X), Y \right)$$

Note that for $X \in \mathfrak{X}(M)$ as the vertical lift $X^v \in \ker d\pi$, we have $d\pi(X^v) = 0$ that is $df(d\pi(X^v)) = 0$, which gives $d(\widetilde{\pi} \circ F)(X^v) = 0$ and consequently that $dF(X^v) \in \ker d\widetilde{\pi} = \bar{\mathfrak{V}}$. We have following lemmas expressing the images of the vertical and horizontal lifts of vector fields on M under the differential of F as well as determining a normal vector field to the submanifold TM.

Lemma 2.2 [2] For $P = (p, X_p) \in TM$

$$dF_P(X_P^v) = (df_p(X_p))^v$$

Lemma 2.3 [2] Let N be the unit normal vector field to the orientable hypersurface M of the Euclidean space R^{n+1} and $P=(p,X_p)\in TM$. Then the horizontal lift Y_P^h of $Y_p\in T_pM$ satisfies

$$dF_P(Y_P^h)=(df_p(Y_p))^h+V_P$$

where $V_P\in\mathfrak{V}_P$ is given by $V_P=g\left(S_p(X_p),Y_p\right)N_P^v$.

Lemma 2.4 [2] Let $\overline{N}=(N,0)\in\mathfrak{X}(R^{2n+2})$, where N is the unit normal vector field of the orientable hypersurface M in the Euclidean space R^{n+1}. Then $\overline{N}=N^h$ and $\overline{N}$ is a normal vector field to TM as a submanifold of R^{2n+2}.

Next, choose N^* a unit normal vector field to TM in R^{2n+2} which is orthogonal to $\overline{N}$ so that $\left\{\overline{N},N^*\right\}$ is a local orthonormal frame of normals on TM. Then for $X,Y\in\mathfrak{X}(TM)$, the second fundamental form h of the submanifold TM has the following expression

$$h(X,Y)=\left\langle h(X,Y),\overline{N}\right\rangle\overline{N}+\langle h(X,Y),N^*\rangle N^*=\overline{g}\left(\bar{S}_{\overline{N}}X,Y\right)\overline{N}+\overline{g}\left(\bar{S}_{N^*}X,Y\right)N^*.$$

The properties of the unit normals $\overline{N}$, N^* and their covariant derivatives are described in the following:

Lemma 2.5 [2] The unit normal N^* to TM is a vertical vector field on the tangent bundle TR^{n+1}.

Lemma 2.6 [2] For $X\in\mathfrak{X}(M)$ and $\overline{N}=(N,0)\in\mathfrak{X}(R^{2n+2})$ we have

$$\bar{D}_{X^h}\overline{N}=(D_XN)^h\quad\text{and}\quad\bar{D}_{X^v}\overline{N}=0.$$

Corollary 2.1 [2] For $X\in\mathfrak{X}(M)$ we have $(S(X))^h=-\bar{D}_{X^h}\overline{N}$.

Remark: We observe that the metrics defined on TM using the Riemannian metric of M (such as Sasaki metric, Cheeger-Gromoll metric, Oproiu metric) are natural metrics in the sense that the projection $\pi:TM\to M$ becomes a Riemannian submersion with respect to these metrics. However, the induced metric on the tangent bundle TM of a hypersurface M of the Euclidean space R^{n+1}, as a submanifold of R^{2n+2} is not a natural metric because of the presence of the term V_P in Lemma 2.3. Indeed the vertical lift N^v of the unit normal vector field N to the tangent bundle R^{2n+2} is tangent to the submanifold TM (cf. Lemma 2.3) and this vector field plays an important role in the study of the geometry of the tangent bundle TM of the hypersurface M of the Euclidean space R^{n+1}.

3. TANGENT BUNDLE OF THE HYPERSURFACE

Let M be an orientable hypersurface of the Euclidean space R^{n+1} with immersion $f : M \to R^{n+1}$ and TM be its tangent bundle with immersion $F : TM \to R^{2n+2}$. We denote the induced metrics on M, TM by g, $\overline{g}$ respectively and the Euclidean metric on R^{n+1} as well as that on R^{2n+2} by $\langle , \rangle$. We also denote by ∇, $\overline{\nabla}$, D and $\overline{D}$ the Riemannian connections on M, TM, R^{n+1}, and R^{2n+2} respectively. Let N and S be the unit normal vector field and the shape operator of the hypersurface M. Then using Lemmas 2.2 and 2.3, we immediately have the following:

Lemma 3.1. If M is an orientable hypersurface of the Euclidean space R^{n+1}, and TM is its tangent bundle as submanifold of R^{2n+2}, then the metric $\bar{g}$ on TM for $P = (p.u) \in TM$, satisfies:
(i) $\bar{g}_P(X_P^h, Y_P^h) = g_p(X_p, Y_p) + g_p(S_p(X_p), u)\, g_p(S_p(Y_p), u)$
(ii) $\bar{g}_P(X_P^h, Y_P^v) = 0$
(iii) $\bar{g}_P(X_P^v, Y_P^v) = g_p(X_p, Y_p)$

In what follows, we will drop the suffixes and it will be understood from the context of the entities appearing in the equation. Now we prove the following:

Theorem 3.1. Let M be an orientable hypersurface of the Euclidean space R^{n+1}, and TM be its tangent bundle as submanifold of R^{2n+2}. Then
(i) $\bar{\nabla}_{X^h} Y^h = (\nabla_X Y)^h - \frac{1}{2}(R(X,Y)u)^v$
(ii) $\bar{\nabla}_{X^h} Y^v = (\nabla_X Y)^v + g(S(X), Y) \circ \pi\ N^v$
(iii) $\bar{\nabla}_{X^v} Y^h = g(S(X), Y) \circ \pi\ N^v$
(iv) $\bar{\nabla}_{X^v} Y^v = 0$

Proof. We use the Kozul's formula

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_{X^h} Y^h, Z^h) &= X^h \bar{g}(Y^h, Z^h) + Y^h \bar{g}(Z^h, X^h) - Z^h \bar{g}(X^h, Y^h) \\ &\quad - \bar{g}(X^h, [Y^h, Z^h]) + \bar{g}(Y^h, [Z^h, X^h]) + \bar{g}(Z^h, [X^h, Y^h]) \end{aligned}$$

and Lemma 3.1, to get

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_{X^h} Y^h, Z^h) &= X^h [g(Y,Z) + g(S(Y),u)\, g(S(Z),u)] \circ \pi \\ &\quad + Y^h [g(Z,X) + g(S(Z),u)\, g(S(X),u)] \circ \pi \\ &\quad - Z^h [g(X,Y) + g(S(X),u)\, g(S(Y),u)] \circ \pi \ - \bar{g}(X^h, [Y,Z]^h) \\ &\quad + \bar{g}(Y^h, [Z,X]^h) + \bar{g}(Z^h, [X,Y]^h) \\ &= X^h g(Y,Z) \circ \pi + Y^h g(Z,X) \circ \pi - Z^h g(X,Y) \circ \pi \\ &\quad + X^h (g(S(Y),u) \circ \pi) g(S(Z),u) \circ \pi \\ &\quad + Y^h (g(S(Z),u) \circ \pi) g(S(X),u) \circ \pi \\ &\quad + g(S(Z),u) \circ \pi Y^h (g(S(X),u) \circ \pi) \\ &\quad - Z^h (g(S(X),u) \circ \pi) g(S(Y),u) \circ \pi \\ &\quad - g(S(X),u) \circ \pi Z^h (g(S(Y),u) \circ \pi) - \bar{g}(X^h, [Y,Z]^h) \\ &\quad + \bar{g}(Y^h, [Z,X]^h) + \bar{g}(Z^h, [X,Y]^h). \end{aligned}$$

Now, using the facts $X^h\overline{g}\,(Y,u)\circ\pi) = g\,(\nabla_X Y,u)\circ\pi,\ X^h((Y,e_i)\circ\pi) = g\,(\nabla_X Y,e_i)\circ\pi$ and that for each $p\in M$ the shape operator gives $S_p : T_pM \to T_pM$ a linear map, choosing a basis $\{e_1,..,e_n\}$ of T_pM that diagonalizes S_p with $S_p(e_i)=\lambda_i e_i$, we get

$$\begin{aligned}
X^h\overline{g}\,(Y,Su)\circ\pi) &= X^h\left(\overline{g}\left(Y,S\left(\sum g_p(u,e_i)e_i\right)\right)\circ\pi\right)\\
&= \sum \lambda_i g_p(u,e_i)X^h\left(\overline{g}\,(Y,e_i)\circ\pi\right)\\
&= \sum \lambda_i g_p(u,e_i)g\,(\nabla_X Y,e_i)\circ\pi = g\,(\nabla_X Y,Su)\circ\pi
\end{aligned}$$

Consequently, using above equation and

$$X^h(\bar{g}(Y^v,Z^v) = \bar{g}((\nabla_X Y)^v,Z^v)+\bar{g}(Y^v,(\nabla_X Z)^v)$$

we arrive at

$$\begin{aligned}
2\bar{g}(\bar{\nabla}_{X^h}Y^h,Z^h) &= \bar{g}((\nabla_X Y)^v,Z^v)+\bar{g}(Y^v,(\nabla_X Z)^v)+\bar{g}((\nabla_Y Z)^v,X^v)\\
&\quad+\bar{g}(Z^v,(\nabla_Y X)^v)-\bar{g}((\nabla_Z X)^v,Y^v)-\bar{g}(X^v,(\nabla_Z Y)^v)\\
&\quad+g\,(S(Z),u)\,g\,(S(\nabla_X Y),u)\circ\pi+g\,(S(Y),u)\,g\,(S(\nabla_X Z),u)\circ\pi\\
&\quad+g\,(S(X),u)\,g\,(S(\nabla_Y Z),u)\circ\pi+g\,(S(Z),u)\,g\,(S(\nabla_Y X),u)\circ\pi\\
&\quad-g\,(S(Y),u)\,g\,(S(\nabla_Z X),u)\circ\pi-g\,(S(X),u)\,g\,(S(\nabla_Z Y),u)\circ\pi\\
&\quad-\bar{g}(X^h,[Y,Z]^h)+\bar{g}(Y^h,[Z,X]^h)+\bar{g}(Z^h,[X,Y]^h),
\end{aligned}$$

that is

$$\begin{aligned}
2\bar{g}(\bar{\nabla}_{X^h}Y^h,Z^h) &= \bar{g}((\nabla_X Y)^v,Z^v)+\bar{g}((\nabla_Y X)^v,Z^v)+\bar{g}((\nabla_X Z)^v,Y^v)\\
&\quad-\bar{g}((\nabla_Z X)^v,Y^v)+\bar{g}((\nabla_Y Z)^v,X^v)-\bar{g}((\nabla_Z Y)^v,X^v)\\
&\quad+g\,(S(Z),u)\,[g\,(S(\nabla_X Y),u)\circ\pi+g\,(S(\nabla_Y X),u)\circ\pi]\\
&\quad+g\,(S(Y),u)\,[g\,(S(\nabla_X Z),u)\circ\pi-g\,(S(\nabla_Z X),u)\circ\pi]\\
&\quad+g\,(S(X),u)\,[g\,(S(\nabla_Y Z),u)\circ\pi+g\,(S(\nabla_Z Y),u)\circ\pi]\\
&\quad-\bar{g}(X^h,[Y,Z]^h)+\bar{g}(Y^h,[Z,X]^h)+\bar{g}(Z^h,[X,Y]^h)
\end{aligned}$$

Thus we have

$$\begin{aligned}
2\bar{g}(\bar{\nabla}_{X^h}Y^h,Z^h) &= \bar{g}((\nabla_X Y)^v+(\nabla_Y X)^v,Z^v)+\bar{g}([X,Z]^v,Y^v)+\bar{g}([Y,Z]\,,X^v)\\
&\quad+g\,(S(Z),u)\,g\,(S(\nabla_X Y)+S(\nabla_Y X),u)\circ\pi\\
&\quad+g\,(S(Y),u)\,g\,(S\,[X,Z]\,,u)\circ\pi\\
&\quad+g\,(S(X),u)\,g\,(S\,[Y,Z]\,,u)\circ\pi-g\,(X,[Y,Z])\circ\pi\\
&\quad-g\,(S(X),u)\,g\,(S\,[Y,Z]\,,u)\circ\pi\ -g\,(Y,[X,Z])\circ\pi\\
&\quad-g\,(S(Y),u)\,g\,(S\,[X,Z]\,,u)\circ\pi\\
&\quad-g\,(S(Y),u)\,g\,(S\,[X,Z]\,,u)\circ\pi
\end{aligned}$$

Using Lemma 3.1 we get

$$2\bar{g}(\bar{\nabla}_{X^h}Y^h, Z^h) = 2g\,(\nabla_X Y, Z)\circ\pi + 2g\,(S(Z), u)\,g\,(S(\nabla_X Y), u)\circ\pi = 2\bar{g}((\nabla_X Y)^h, Z^h),$$

that is

$$\bar{g}(\bar{\nabla}_{X^h}Y^h, Z^h) = \bar{g}((\nabla_X Y)^h, Z^h) \tag{3.1}$$

On the other hand, using the fact that

$$X^h(\bar{g}(Y^v, Z^v) = \bar{g}((\nabla_X Y)^v, Z^v) + \bar{g}(Y^v, (\nabla_X Z)^v)$$

and the Kozul's formula we have

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_{X^h}Y^h, Z^v) &= X^h\cdot\bar{g}(Y^h, Z^v) + Y^h\cdot\bar{g}(Z^v, X^h) - Z^v\cdot\bar{g}(X^h, Y^h) \\ &\quad -\bar{g}(X^h, \left[Y^h, Z^v\right]) + \bar{g}(Z^v, \left[X^h, Y^h\right]) \\ &= -\bar{g}(X^h, (\nabla_Y Z)^v) + \bar{g}(Y^h, -(\nabla_X Z)^v) \\ &\quad +\bar{g}(Z^v, [X, Y]^h - (R(X, Y)u)^v) \\ &= -\bar{g}(Z^v, (R(X, Y)u)^v), \end{aligned}$$

which together with proposition 5.1 in [5] gives

$$\bar{g}(\bar{\nabla}_{X^h}Y^h, Z^v) = -\frac{1}{2}\bar{g}((R(X, Y)u)^v, Z^v)$$

Combining this last equation with equation (3.1) we get (i).

To prove (ii), we use the immersion $F : TM \to R^{2n+2}$ to write the Gauss equation (2.2) in the form

$$\bar{D}_{dF(X^h)}dF(Y^v) = dF(\bar{\nabla}_{X^h}Y^v) + h(X^h, Y^v).$$

Then the Lemma 2.3 gives

$$\bar{D}_{[df(X)]^h+V}\,[df(Y)]^v = dF(\bar{\nabla}_{X^h}Y^v) + h(X^h, Y^v)$$

Note that the metric on the tangent bundle $TR^{n+1} = R^{2n+2}$ of R^{n+1} being Sasaki metric, using the corresponding equation in [5], we get

$$\left[D_{df(X)}df(Y)\right]^v = dF(\bar{\nabla}_{X^h}Y^v) + h(X^h, Y^v)$$

that is

$$[df(\nabla_X Y) + g\,(S(X), Y)\,N]^v = dF(\bar{\nabla}_{X^h}Y^v) + h(X^h, Y^v)$$

Since N^v is tangent to TM, equating the tangential and normal components we get

$$dF(\bar{\nabla}_{X^h}Y^v) = dF((\nabla_X Y)^v) + (S(X), Y)\circ\pi N^v, \quad h(X^h, Y^v) = 0 \tag{3.2}$$

that gives

$$\bar{\nabla}_{X^h}Y^v = (\nabla_X Y)^v + (S(X), Y) \circ \pi N^v$$

which proves (ii).

For (iii), we use the fact that $\left[Y^h, X^v\right] = (\nabla_Y X)^v$ (cf. [5], proposition 5.1) and (ii) to get

$$(\nabla_Y X)^v + g\left(S(Y), X\right) \circ \pi N^v - \bar{\nabla}_{X^v}Y^h = (\nabla_Y X)^v$$

which proves (iii).

Finally for (iv), we use the Kozul's formula and proposition 5.1 in [5] together with Lemma 3.1, to get

$$\begin{aligned} 2\bar{g}\ (\bar{\nabla}_{X^v}Y^v, Z^h) &= -Z^h g\left(X, Y\right) \circ \pi - \bar{g}(X^v, -(\nabla_Z Y)^v) + \bar{g}(Y^v, (\nabla_Z X)^v) \\ &= -Zg\left(X, Y\right) \circ \pi + g(X, (\nabla_Z Y)) \circ \pi + g(Y, (\nabla_Z X)) \circ \pi \\ &= 0 \end{aligned}$$

and

$$2\bar{g}\ (\bar{\nabla}_{X^v}Y^v, Z^v) = X^v g\left(Y, Z\right) \circ \pi + Y^v g\left(Z, X\right) \circ \pi - Z^v g\left(X, Y\right) \circ \pi = 0$$

which proves (iv).

Lemma 3.2. Let M be an orientable hypersurface of the Euclidean space R^{n+1}. Then for $X, Y \in \mathfrak{X}(M)$, the second fundamental form of the submanifold TM satisfies
(i) $h(X^v, Y^v) = 0$
(ii) $h(X^v, Y^h) = 0$
(iii) $h(X^h, Y^h) = g\left(S(X), Y\right) \circ \pi\ N^h$

Proof. The statement (ii) is already proved in equation (3.2). For (i), we have from equation (2.3) for the submanifold TM that $\bar{D}_{X^v}Y^v = \bar{\nabla}_{X^v}Y^v + h(X^v, Y^v)$ which together with the fact that metric on the tangent bundle TR^{n+1} is a Sasaki metric, that is, $\bar{D}_{X^v}Y^v = 0$, we get $h(X^v, Y^v) = 0$.

For (iii), we have local orthonormal unit normal vector fields $\overline{N} = N^h$, N^* for the submanifold TM as described in Lemmas 2.4 and 2.5, where N^* is vertical on the tangent bundle TR^{n+1}. Then using equation (2.3) as $\bar{D}_{dF(X^h)}dF(Y^h) = \bar{\nabla}_{X^h}Y^h + h(X^h, Y^h)$ together with Lemma 2.3 and Proposition 7.2 in [5] for Sasaki metric on TR^{n+1} we arrive at

$$\begin{aligned} \left\langle h(X^h, Y^h), \overline{N}\right\rangle &= \left\langle \bar{D}_{dF(X^h)}dF\left(Y^h\right), \overline{N}\right\rangle = -\left\langle\ \bar{D}_{dF(X^h)}\overline{N}, dF(Y^h)\right\rangle \\ &= -\left\langle\ \left(D_{dF(X)}N\right)^h, dF(Y^h)\right\rangle = -\left\langle\ \left(D_{dF(X)}N\right)^h, (dF(Y))^h\right\rangle \\ &= -\left\langle\ \left(D_{dF(X)}N\right), (dF(Y))\right\rangle \circ \overline{\pi} = g(SX, Y) \circ \pi. \end{aligned} \tag{3.3}$$

Now, we use the facts that N^v is tangent to TM and the unit normal vector field N^* is vertical on the tangent bundle TR^{n+1} together with Lemma 2.3 and

Proposition 7.2 in [5], for Sasaki metric on the tangent bundle TR^{n+1} to get

$$\begin{aligned}\left\langle h(X^h,Y^h),N^*\right\rangle &= \left\langle \bar{D}_{dF(X^h)}dF\left(Y^h\right),N^*\right\rangle \\ &= \left\langle \bar{D}_{(dF(X))^h+g(SX,u)\circ\pi N^v}\left(dF\left(Y\right)\right)^h+g(SY,u)\circ\pi N^v,N^*\right\rangle \\ &= \left\langle \left(D_{dF(X)}dF\left(Y\right)\right)^h+\left(Xg(SY,u)\right)\circ\pi N^v,N^*\right\rangle \\ &\quad +\left\langle g(SY,u)\circ\pi\left(D_{dF(X)}N\right)^v,N^*\right\rangle \\ &= -g(SY,u)\circ\pi\left\langle (SX)^v,N^*\right\rangle=0\end{aligned}$$

where we used $(SX)^v\in\mathfrak{X}(TM)$ and that N^* is normal vector field to the submanifold TM. Combining above equation with equation (3.3) we get

$$h(X^h,Y^h)=g\left(S(X),Y\right)\circ\pi N^h$$

Lemma 3.3. For an orientable hypersurface M of the Euclidean space $R^{n+1}$and $X\in\mathfrak{X}(M)$, the covariant derivatives of the vertical and horizontal lifts of the unit normal vector field N are given by
(i) $\bar{D}_{X^v}N^v=0$
(ii) $\bar{D}_{X^v}N^h=0$
(iii) $\bar{D}_{X^h}N^v=-(S(X))^v$
(iv) $\bar{D}_{X^h}N^h=-(S(X))^h$

Proof. We express the unit normal vector field N to the hypersurface M locally as

$$N=h^i\frac{\partial}{\partial u^i}$$

for smooth functions h^i on R^{n+1}. Consequently its vertical lift N^v and the horizontal lift N^h have the expressions

$$N^v=(h^i\circ\pi)\frac{\partial}{\partial v^i},\quad N^h=(h^i\circ\pi)\frac{\partial}{\partial u^i}.$$

Since the Christoffel symbols $\Gamma^j_{ik}=0$ for the Euclidean connection D on R^{n+1}, using the properties of Euclidean connection $\overline{D}$ on $TR^{n+1}=R^{2n+2}$, we have

$$\bar{D}_{X^v}N^v=(df(X))^v\ (h^i\circ\pi)\frac{\partial}{\partial v^i}=0$$

$$\bar{D}_{X^v}N^h=(df(X))^v\ (h^i\circ\pi)\frac{\partial}{\partial u^i}=0$$

$$\begin{aligned}\bar{D}_{X^h}N^v &= dF(X^h)(\ (h^i\circ\pi)\frac{\partial}{\partial v^i}) \\ &= [(df(X))^h+g(SX,u)\circ\pi N^v](\ (h^i\circ\pi)\frac{\partial}{\partial v^i}) \\ &= (df(X))^h\ (h^i\circ\pi)\frac{\partial}{\partial v^i}=(df(X))(h^i)\circ\pi\frac{\partial}{\partial v^i}.\end{aligned}$$

On the other hand, we have $D_X N = \ (df(X))(h^i)\frac{\partial}{\partial u^i}$, which gives

$$(D_X N)^v = \ (df(X))(h^i) \circ \pi \frac{\partial}{\partial v^i}$$

Combining these last two equations with equation (2.2) we get (iii).

Finally, note that

$$\begin{aligned}
\bar{D}_{X^h} N^h &= dF(X^h)(\ (h^i \circ \pi) \frac{\partial}{\partial u^i}) \\
&= [(df(X))^h + g(SX, u) \circ \pi N^v](\ (h^i \circ \pi) \frac{\partial}{\partial u^i}) \\
&= (df(X))^h \ \ (h^i \circ \pi) \frac{\partial}{\partial u^i}.
\end{aligned}$$

and $D_X N = \ (df(X))(h^i)\frac{\partial}{\partial u^i}$, to conclude

$$(D_X N)^h = \ (df(X))(h^i) \circ \pi \frac{\partial}{\partial u^i}$$

and these two equations together with equation (2.2) prove (iv).

Lemma 3.4. For the tangent bundle TM of an orientable hypersurface M of the Euclidean space R^{n+1} we have

(i) $h(X^v, N^v) = 0$, (ii) $\bar{\nabla}_{X^v} N^v = 0$

(iii) $h(X^h, N^v) = 0$, (iv) $\bar{\nabla}_{X^h} N^v = -(S(X))^v$, $X \in \mathfrak{X}(M)$.

Proof. Using the local expression $N = h^i \frac{\partial}{\partial u^i}$, we get the following expression for its horizontal lift to the tangent bundle TR^{n+1}

$$\bar{N} = N^h = (h^i \circ \pi) \frac{\partial}{\partial u^i} \ ,$$

which is a unit normal vector field to the submanifold TM. As the metric on the tangent bundle TR^{n+1} is the Sasaki metric and N^v is tangent to TM, we have

$$0 = \bar{D}_{X^v} N^v = \bar{\nabla}_{X^v} N^v + h(X^v, N^v),$$

which on equating tangential and normal components proves (i) and (ii).

Now using Lemma 2.3, we have

$$\begin{aligned}
\bar{D}_{dF(X^h)} N^v &= \bar{D}_{(df(X))^h + g(S(X),u) \circ \pi N^v} N^v \\
&= \bar{D}_{(df(X))^h} N^v + g\,(S(X), u) \circ \pi \bar{D}_{N^v} N^v \\
&= -(S(X))^v.
\end{aligned}$$

Also, as N^v is tangent to the submanifold TM, we use equation (2.3) to get

$$-(S(X))^v = \bar{D}_{dF(X^h)} N^v = (\bar{\nabla}_{X^h} N^v) + h(X^h, N^v)$$

and on equating the tangential and normal components we get (iii) and (iv).

Now we are in a position to prove the following codimension reduction Theorem.

Theorem 3.2. Let M be an orientable hypersurface of the Euclidean space R^{n+1} with unit normal vector field N. Then the tangent bundle TM is a hypersurface of the Euclidean space R^{2n+1} with unit normal vector field N^h the horizontal lift of N with respect to the natural submersion $\pi : R^{2n+1} \to R^{n+1}$, $\pi(u^1, .., u^{2n+1}) = (u^1, .., u^{n+1})$.

Proof: Using Lemmas 2.4 and 2.5, we observe that for each $P \in TM$, the set $\left\{N_P^h, N_P^*\right\}$ is an orthonormal basis of the normal space $T_P^{\perp}TM$. We denote by and $\bar{S}_{\hat{N}}$ the Weingarten map in the direction of the normal $\hat{N}$ to the submanifold TM of the Euclidean space R^{2n+2}. Then by Lemmas 3.2 and 3.4, we observe that $\bar{S}_{N^*}(X^h) = \bar{S}_{N^*}(X^v) = \bar{S}_{N^*}(N^v) = 0$ for each $X \in \mathfrak{X}(M)$, that is $\bar{S}_{N^*}(E) = 0$ for each $E \in \mathfrak{X}(TM)$, consequently we get that the first normal space $N_1(P)$, $P \in TM$ (cf. [4]) is spanned by the unit normal vector N_P^h. Using the local expression of the unit normal vector field N to the hypersurface M

$$N = h^i \frac{\partial}{\partial u^i},$$

we get the following local expression for the horizontal lift N^h

$$N^h = (h^i \circ \pi) \frac{\partial}{\partial u^i},$$

which gives

$$\bar{D}_{N^v} N^h = N^v(h^i \circ \pi) \frac{\partial}{\partial u^i} = 0$$

and consequently

$$\nabla^{\perp}_{N^v} N^h = 0, \tag{3.4}$$

where $\nabla^{\perp}$ is the connection in the normal bundle of the submanifold TM of the Euclidean space R^{2n+2}. Moreover by Lemma 3.3 together with equation (2.4), we get

$$\nabla^{\perp}_{X^v} N^h = 0, \quad \nabla^{\perp}_{X^h} N^h = 0, \quad X \in \mathfrak{X}(M) \tag{3.5}$$

The equations (3.4) and (3.5) imply that the first normal spaces $N_1(P)$ are invariant under the parallel translation with respect to the normal connection $\nabla^{\perp}$. Since the dimension of $N_1^{\perp}(P)$, the orthogonal complement of the first normal space $N_1(P)$ in $T_P^{\perp}TM$, is 1 for each $P \in TM$, by the main Theorem in [4], we get that $(TM, \overline{g})$ admits an isometric immersion in the totally geodesic submanifold R^{2n+1} of R^{2n+2}.

4. SECTIONAL CURVATURES OF THE TANGENT BUNDLE

In this section we obtain expressions for the sectional curvatures of the tangent bundle TM of the hypersurface M in a Euclidean space R^{n+1} with unit normal N, as well as study the properties of the vector field N^v. The tangent bundle TM is now a hypersurface of the Euclidean space R^{2n+1} with unit normal vector field N^h (cf. Theorem 3.2). We denote the shape operator of the hypersurface TM by $\overline{S}$, consequently we have

$$\overline{D}_E F = \overline{\nabla}_E F + \overline{g}\left(\overline{S}(E), F\right) N^h, \quad \overline{D}_E N^h = -\overline{S}(E), \quad E, F \in \mathfrak{X}(TM) \quad (4.1)$$

Since, $\overline{D}_{N^v} N^h = 0$, it follows that $\overline{S}(N^v) = 0$ and by Lemma 3.3 that

$$\overline{S}(X^v) = 0, \overline{S}(X^h) = (S(X))^h \quad (4.2)$$

holds. Then using the Gauss equation expressing curvature tensor field for hypersurface TM in the Euclidean space R^{2n+1} together with Lemma 3.1, we immediately have the following:

Theorem 4.1. Let M be an orientable hypersurface of the Euclidean space R^{n+1} and $\tilde{R}$ be the Riemannian curvature tensor field of the tangent bundle $(TM, \bar{g})$ equipped with the induced metric $\bar{g}$ as a hypersurface of R^{2n+1}. Then the following hold:

(i) $\tilde{R}(X^h, Y^h; Z^h, W^h) = \{g(S(Y), Z)g(S(X), W) - g(S(X), Z)g(S(Y), W)\} \circ \pi$

(ii) $\tilde{R}(X^h, Y^h; Z^h, W^v) = 0$

(iii) $\tilde{R}(X^h, Y^h; Z^v, W^v) = 0$

(iv) $\tilde{R}(X^v, Y^v; Z^v, W^v) = 0$

(v) $\tilde{R}(X^v, Y^v; Z^v, W^h) = 0$

(vi) $\tilde{R}(X^h, Y^h; Z^h, N^v) = 0$

(vii) $\tilde{R}(X^h, Y^h; Z^v, N^v) = 0$

(viii) $\tilde{R}(X^v, Y^v; Z^h, N^v) = 0$

(ix) $\tilde{R}(X^v, Y^v; Z^v, W^v) = 0, \quad X, Y, Z, W \in \mathfrak{X}(M)$.

Corollary 4.1 The tangent bundle TS^n of the unit sphere S^n admits a Riemannian metric of non-negative sectional curvature.

Proof: The shape operator of the unit sphere S^n with respect to the natural imbedding of S^n in R^{n+1} is $S = -I$. Consequently, using above Theorem together with an orthonormal set $\{X, Y\}$ of vector fields on S^n, the sectional curvatures of the hypersurface TS^n are given by

$$\tilde{K}(X^v, Y^v) = \frac{\tilde{R}(X^v, Y^v; Y^v, X^v)}{\|X^v \wedge Y^v\|^2} = 0$$

$$\tilde{K}(X^h, Y^v) = \frac{\tilde{R}(X^h, Y^v; Y^v, X^h)}{\|X^h \wedge Y^v\|^2} = 0$$

$$\tilde{K}(X^h, Y^h) = \frac{\tilde{R}(X^h, Y^h; Y^h, X^h)}{\|X^h \wedge Y^h\|^2} = \frac{1}{\|X^h \wedge Y^h\|^2}$$

where by Lemma 3.1, $\left\|X^h \wedge Y^h\right\|_P^2 = \bar{g}_P(X_P^h, X_P^h)\bar{g}_P(Y_P^h, Y_P^h) - \bar{g}_P(X_P^h, Y_P^h)^2 = 1 + \langle X_p, u\rangle^2 + \langle Y_p, u\rangle^2$, with $P = (p, u) \in TS^n$. Also we have

$$\tilde{K}(X^v, N^v) = \frac{\tilde{R}(X^v, N^v; N^v, X^v)}{\|X^v \wedge N^v\|^2} = 0$$

$$\tilde{K}(X^h, N^v) = \frac{\tilde{R}(X^h, N^v; N^v, X^h)}{\|X^h \wedge N^v\|^2} = 0$$

and this completes the proof.

In the rest of this section we shall study the properties of the vector field N^v defined on the tangent bundle TM of the hypersurface M of the Euclidean space R^{n+1}. First we recall that a geodesic $\sigma(t)$ in the tangent bundle TM is said to be horizontal (vertical) if the tangent vector field $\dot{\sigma}(t)$ is horizontal (vertical). We have the following theorem which is an immediate consequence of equation (4.2) and the Gauss equation expressing the curvature tensor field of the hypersurface TM.

Theorem 4.2. Let M be an orientable hypersurface of the Euclidean space R^{n+1} with unit normal vector field N. Then the vector field N^v on the tangent bundle TM is a Jacobi field along any vertical geodesic of TM.

It is well know than if the tangent bundle TM of a smooth manifold M is trivial, then its Euler characteristic class $\chi(M) = 0$, and it is an interesting question to obtain conditions under which the tangent bundle is trivial. In the following theorem, we see that the vector field N^v plays an important role in predicting when the tangent bundle of an orientable hypersurface of the Euclidean space R^{n+1} is trivial

Theorem 4.2. Let M be an orientable hypersurface of the Euclidean space R^{n+1} with unit normal vector field N. If the deRham cohomology group $H^1(TM, R) = 0$, then the tangent bundle TM is trivial.

Proof: Define a smooth 1-form α on the tangent bundle TM by $\alpha(E) = \overline{g}(E, N^v)$, $E \in \mathfrak{X}(TM)$. Then as TM is hypersurface of the Euclidean space R^{2n+1} with unit normal vector field N^h and shape operator $\overline{S}$, the formulas in Lemma 3.4 imply that

$$\overline{\nabla}_{X^V} N^v = 0, \quad \overline{\nabla}_{X^h} N^v = -\left(S(X)\right)^v, \quad X \in \mathfrak{X}(M) \tag{4.3}$$

where S is the shape operator of the hypersurface M. Since $T_PTM = \mathfrak{H}_P \oplus \mathfrak{V}_P$, $P \in TM$, Using equations in (4.3), we conclude that

$$d\alpha\left(X^v, Y^v\right) = 0, \quad d\alpha(X^h, Y^v) = 0, \quad d\alpha\left(X^h, Y^h\right) = 0$$

Since $T_PTM = \mathfrak{H}_P \oplus \mathfrak{V}_P$, $P \in TM$, it follows that the form α is closed. However, as $H^1(TM, R) = 0$, the form α has to be exact and consequently there is a smooth function $\varphi \in C^\infty(TM)$ such that $N^v = grad\varphi$, where $grad\varphi$ is the gradient of φ on the Riemannian manifold $(TM, \overline{g})$. The Hessian form H_φ is given by

$$H_\varphi(E, F) = \overline{g}(\overline{\nabla}_E grad\varphi, F) = \overline{g}\left(\overline{\nabla}_E N^v, F\right), \quad E, F \in \mathfrak{X}(TM)$$

which together with equation (4.3) gives,

$$H_\varphi(X^h, Y^v) = -\overline{g}(\left(S(X)\right)^v, Y^v), \quad H_\varphi(Y^v, X^h) = 0, \quad X, Y \in \mathfrak{X}(M)$$

However, the Hessian operator H_φ being symmetric we get $\left(S(X)\right)^v = 0$, for each $X \in \mathfrak{X}(M)$. Thus the Gauss equation for the hypersurface M of R^{n+1} gives $\left[R(X, Y)Z\right]^v = 0$. Using this in Theorem 3.1, we conclude that

$$\bar{\nabla}_{X^h} Y^h = (\nabla_X Y)^h \tag{4.4}$$

This proves that for $X^h, Y^h \in \mathfrak{H}$, $[X^h, Y^h] \in \mathfrak{H}$, that is the horizontal distribution $\mathfrak{H}$ is integrable. Moreover, by equation (4.4) it follows that the leaves of the distribution $\mathfrak{H}$ are totally geodesic in TM. Also, by (iv) in Theorem 3.1, it follows that the vertical distribution $\mathfrak{V}$ is also integrable with leaves totally geodesic in TM. Note that the equation (4.2) implies that the leaves of $\mathfrak{V}$ are totally geodesic submanifolds of R^{2n+1} and consequently are isometric to R^n. The projection map $\pi : TM \to M$ restricted to a leaf of $\mathfrak{H}$ is a diffeomorphism. This proves that TM is diffeomorphic to $M \times R^n$, that is the tangent bundle TM is trivial.

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