SPHERICAL SUBMANIFOLDS OF A EUCLIDEAN SPACE

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1. Introduction

An interesting quest in geometry is to obtain characterizations of spherical curves. There are many characterizations for a curve in $\mathbb{R}^3$ to lie on the unit sphere S^2. One is due to Breuer and Gottlieb [**1**] (see also [**12**]), who have shown that a curve is spherical if and only if the radius of curvature $\rho(s)$ and the torsion $\tau(s)$ satisfy the explicit relation

$$\rho(s) = A_1 \cos\left[\int \tau(s)\,\mathrm{d}s\right] + A_2 \sin\left[\int \tau(s)\,\mathrm{d}s\right],$$

A_1 and A_2 being arbitrary constants.

A natural generalization of this question to higher dimensions could be: 'given an isometric immersion $\psi : M^n \to \mathbb{R}^{n+2}$ of a compact n-dimensional Riemannian manifold (M^n, g), obtain conditions for $\psi(M^n) \subset S^{n+1}(c)$, where $S^{n+1}(c)$ is the sphere of constant curvature c'. We write ψ^T, $\psi^\perp$ as tangential and normal components of the position vector ψ in $\mathbb{R}^{n+p}$ and show that for a connected submanifold $\psi : M^n \to \mathbb{R}^{n+p}$, $\psi(M^n) \subset S^{n+p-1}(c)$ for some c if and only if $\langle H, \psi^\perp\rangle$ is a constant, where H is the mean curvature vector field (Theorem 4.1). Similarly it is observed that for this submanifold the vector field ψ^T being harmonic also provides a necessary and sufficient condition for $\psi(M^n) \subset S^{n+p-1}(c)$, for some constant c (Theorem 4.3).

We also study codimension-2 isometric immersions $\psi : M^n \to \mathbb{R}^{n+2}$ of a compact connected Riemannian manifold (M^n, g) with parallel mean curvature vector field and observe that if the sectional curvatures of M^n are strictly positive and the scalar curvature S of M^n satisfies $S < n(n-1)\lambda^{-2}$, then no such isometric immersion of (M^n, g) is contained in a ball of radius λ in $\mathbb{R}^{n+2}$ (Theorem 5.1).

2. Preliminaries

Let M^n be an immersed submanifold of the Euclidean space $\mathbb{R}^{n+p}$. We denote by $\langle , \rangle$ the inner product on $\mathbb{R}^{n+p}$, by g the induced Riemannian metric on M^n, by $\mathfrak{X}(M^n)$ the Lie algebra of smooth vector fields on M^n, by $\Gamma(\nu)$ the space of sections of the normal bundle ν of M^n and by $\overline{\nabla}$ and ∇ the Riemannian connections on $\mathbb{R}^{n+p}$ and on M^n and respectively. Then we have (for details we refer to [**3**])

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \overline{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.1}$$

$X, Y \in \mathfrak{X}(M^n)$, $N \in \Gamma(\nu)$, where h is the second fundamental form, $\nabla^\perp$ is the normal connection and A_N is the Weingarten map corresponding to the normal section $N \in \Gamma(\nu)$ which satisfies

$$g(A_N X, Y) = \langle h(X, Y), N\rangle, \quad X, Y \in \mathfrak{X}(M^n), N \in \Gamma(\nu). \tag{2.2}$$

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The equations of Gauss, Codazzi and Ricci are respectively

$$R(X, Y)Z = A_{h(Y,Z)}X - A_{h(X,Z)}Y, \tag{2.3}$$

$$(\nabla h)(X, Y, Z) = (\nabla h)(Y, Z, X), \tag{2.4}$$

$$R^{\perp}(X, Y, N_1, N_2) = g([A_{N_I}, A_{N_2}](X), Y) \tag{2.5}$$

for $X, Y, Z \in \mathfrak{X}(M^n)$, and $N_1, N_2 \in \Gamma(\nu)$, where R, $R^{\perp}$ are the curvature tensors corresponding to the connections ∇ and $\nabla^{\perp}$ respectively. The submanifold M^n is said to have flat normal connection if $R^{\perp} = 0$. The covariant derivative $(\nabla h)(X, Y, Z)$ is given by $(\nabla h)(X, Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$ and the covariant derivatives of a (1, 1) tensor field A on M^n are given by

$$(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y)$$

and

$$(\nabla^2 A)(X, Y, Z) = \nabla_X(\nabla A)(Y, Z) - (\nabla A)(\nabla_X Y, Z) - (\nabla A)(Y, \nabla_X Z)$$

respectively, and we have the Ricci identity

$$(\nabla^2 A)(X, Y, Z) - (\nabla^2 A)(Y, X, Z) = R(X, Y)AZ - AR(X, Y)Z. \tag{2.6}$$

Let Ric be the Ricci tensor of M^n. Then the Ricci operator Q is a symmetric operator defined by $\mathrm{Ric}(X, Y) = g(Q(X), Y)$, $X, Y \in \mathfrak{X}(M^n)$. The Gauss equation (2.3) gives the following expression for the Ricci operator Q of the submanifold M^n:

$$Q(X) = nA_H X - \sum_i A_{h(e_i, X)} e_i, \tag{2.7}$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M^n and $H = 1/n \sum_i h(e_i, e_i)$ is the mean curvature vector field. The scalar curvature $S = \sum_i \mathrm{Ric}(e_i, e_i)$ of the submanifold is given by

$$S = n^2 \|H\|^2 - \|h\|^2, \tag{2.8}$$

where $\|h\|^2 = \sum_{ij} \|h(e_i, e_j)\|^2$ is the square of the length of the second fundamental form. The mean curvature vector field H is said to be parallel if it satisfies $\nabla^{\perp}_X H = 0$ for all $X \in \mathfrak{X}(M^n)$.

Let $\psi : M^n \to \mathbb{R}^{n+p}$ be the immersion. Then

$$\psi = \psi^T + \psi^{\perp},$$

where $\psi^T \in \mathfrak{X}(M^n)$ and $\psi^{\perp} \in \Gamma(\upsilon)$. Then using (2.1) we get

$$\nabla_X \psi^T = X + A_{\psi^{\perp}} X, \quad \nabla^{\perp}_X \psi^{\perp} = -h(X, \psi^T), \quad X \in \mathfrak{X}(M^n). \tag{2.9}$$

Define a smooth function $F : M^n \to \mathbb{R}$ on the submanifold M^n by $F = \langle H, \psi^{\perp} \rangle$. Then from equation $(2.9)_1$ we obtain $\operatorname{div} \psi^T = n(1 + F)$ and consequently we have the following.

LEMMA 2.1 *Let M^n be a compact oriented submanifold of the Euclidean space $\mathbb{R}^{n+p}$. Then*

$$\int_{M^n} (1+F)\,\mathrm{d}v = 0.$$

REMARK *This lemma is generalization of the well-known Minkowski integral formula for hypersurfaces, as for $p = 1$ we shall have $F = \rho\alpha$, where $\alpha = \|H\|$ is the mean curvature and $\psi^\perp = \rho N$, for N the unit normal vector field.*

For an immersion $\psi : M^n \to \mathbb{R}^{n+p}$, we define a symmetric tensor field B of type (1, 1) by $B(X) = A_{\psi^\perp}(X)$, $X \in \mathfrak{X}(M^n)$. This tensor field has the following properties.

LEMMA 2.2 *The tensor field B satisfies*

(i) $trB = nF$,

(ii) $(\nabla B)(X,Y) - (\nabla B)(Y,X) = R(X,Y)\psi^T$,

(iii) $\sum_i (\nabla B)(e_i, e_i) = n\,\mathrm{grad}\,F + Q(\psi^T)$,

where $\{e_1,\ldots,e_n\}$ is a local orthonormal frame on M^n and

$$(\nabla B)(X,Y) = \nabla_X B(Y) - B(\nabla_X Y),\; X, Y \in \mathfrak{X}(M^n).$$

Proof. Part (i) follows immediately from the definition of B. Since B is symmetric it is easy to verify that

$$g((\nabla B)(X,Y), Z) = g(Y, (\nabla B)(X,Z)) \tag{2.10}$$

holds. Then using equations (2.9) and (2.10) we get after some calculations that $g((\nabla B)(X,Y) - (\nabla B)(Y,X), Z) = g(R(X,Y)\psi^T, Z)$ and this proves (ii). To prove (iii) we have from (i) that $\sum_i g(Be_i, e_i) = nF$. Thus for $X \in \mathfrak{X}(M^n)$, we have

$$\sum_i g((\nabla B)(X, e_i), e_i) = nX(F) = ng(X, \mathrm{grad}\,F).$$

Using (ii) and the definition of the Ricci tensor in above equation we get (iii).

3. Integral formulae

In this section we derive some integral formulae which play a key role in proving our results.

PROPOSITION 3.1 *Let M^n be compact oriented and $\psi : M^n \to \mathbb{R}^{n+p}$ an immersion. Then*

$$\int_{M^n} \{\mathrm{Ric}(\psi^T, \psi^T) + \|B\|^2 - n^2F^2 + n(n-1)\}\,\mathrm{d}v = 0.$$

Proof. Consider a local orthonormal frame $\{e_1,\ldots,e_n\}$ which is covariant constant at a point. Then we use Lemma 2.2 to compute $\mathrm{div}(B\psi^T)$ and arrive at

$$\mathrm{div}(B\psi^T) = nF + \|B\|^2 + ng(\psi^T, \mathrm{grad}\,F) + \mathrm{Ric}(\psi^T, \psi^T). \tag{3.1}$$

We have $\mathrm{div}(F\psi^T) = g(\psi^T, \mathrm{grad}\,F) + F\,\mathrm{div}\,\psi^T$. Using equation $(2.9)_1$ we get $\mathrm{div}\,\psi^T = n(1+F)$. Thus we have $\mathrm{div}(F\psi^T) = g(\psi^T, \mathrm{grad}\,F) + nF + nF^2$. Using this to eliminate the term $g(\psi^T, \mathrm{grad}\,F)$ in (3.1), we arrive at $\mathrm{div}(B\psi^T - nF\psi^T) = -n(n-1)F - n^2F^2 + \|B\|^2 + \mathrm{Ric}(\psi^T, \psi^T)$. Integrating this last equation and using Lemma 2.1 we get the desired result.

PROPOSITION 3.2 *If M^n and ψ are as above and M^n has parallel mean curvature vector field, then*

$$\int_{M^n}\left\{\mathrm{Ric}(\psi^T,\psi^T)-n^2F^2+n^2+\sum_i\|h(e_i,\psi^T)\|^2\right\}\mathrm{d}v=0.$$

Proof. Using equation $(2.9)_2$ and that the mean curvature vector H is parallel in the definition of F, we get $\psi^T(F)=-g(H,h(\psi^T,\psi^T))$. This equation together with (2.7) and the fact that $\mathrm{div}(F\psi^T)=\psi^T(F)+F\,\mathrm{div}\,\psi^T$ gives

$$-\mathrm{div}(nF\psi^T)=\mathrm{Ric}(\psi^T,\psi^T)-n^2F(1+F)+\sum_i\|h(e_i,\psi^T)\|^2,$$

where we have used $\mathrm{div}\,\psi^T=n(1+F)$. Integrating the above equation and using Lemma 2.1 we get the result.

4. Submanifolds lying on a hypersphere

Our first result is the following.

THEOREM 4.1 *Let $\psi:M^n\to\mathbb{R}^{n+p}$ be an immersion of an oriented compact connected manifold. Then $\psi(M^n)\subset S^{n+p-1}(c)$ for some constant $c>0$ if, and only if, the function $F=\langle H,\psi^\perp\rangle$ is a constant.*

Proof. If $\psi(M^n)\subset S^{n+p-1}(c)$, then $\langle\psi,\psi\rangle=c$, and for any $X\in\mathfrak{X}(M^n)$ we get $X\langle\psi,\psi\rangle=2\langle\overline{\nabla}_X\psi,\psi\rangle=2\langle\psi^T,\psi^T\rangle=0$. This gives $\psi^T=0$, and that $\psi=\psi^\perp=N/\sqrt{c}N$, where N is the unit normal to the sphere $S^{n+p-1}(c)$ in $\mathbb{R}^{n+p}$. Thus in this case we have $B=A_{N/\sqrt{c}N}=-I$, and consequently (i) in Lemma 2.2 gives $F=-1$.

Conversely, suppose F is a constant. Then by Lemma 2.1 we get $F=-1$. Now we compute $\mathrm{div}(\frac{1}{2}\|\psi\|^2\psi^T)=\|\psi^T\|^2+\frac{1}{2}\|\psi\|^2\,\mathrm{div}\,\psi^T$. However as $\mathrm{div}\,\psi^T=n(1+F)=0$, integrating we arrive at $\psi^T=0$. Equation $(2.9)_2$ now yields $X(\|\psi^\perp\|^2)=0$, $X\in\mathfrak{X}(M^n)$. Thus $\|\psi\|^2=\|\psi^\perp\|^2=c$, a constant, and proves that $\psi(M^n)\subset S^{n+p-1}(c)$.

In particular, for $p=1$, we have $\psi^\perp=\rho N$, and $H=\alpha N$, where α is the mean curvature. In this case we get the following characterization for spheres in $\mathbb{R}^{n+1}$.

COROLLARY 4.1 *Let $\psi:M^n\to\mathbb{R}^{n+1}$ be an oriented compact and connected hypersurface. Then $\psi(M^n)=S^n(c)$ for some constant $c>0$, if and only if the function $F=\rho\alpha$ is a constant.*

LEMMA 4.1 *Let $\psi:M^n\to\mathbb{R}^{n+p}$ be an oriented, compact and connected submanifold. Then $\psi(M^n)\subset S^{n+p-1}(c)$ for some constant $c>0$ if and only if the function $F=\langle H,\psi^\perp\rangle$ and the Ricci curvature in the direction of ψ^T satisfy*

$$\mathrm{Ric}(\psi^T,\psi^T)\geqslant n^2(1+F)^2.$$

Proof. If $\psi(M^n)\subset S^{n+p-1}(c)$, then by Theorem 4.1 we have $F=-1$ and $\psi^T=0$ and thus the above condition is satisfied. Conversely, suppose that the given condition holds. Note that by Lemma 2.1 we have $\int_{M^n}(1+F)^2\,\mathrm{d}v=\int_{M^n}(F^2-1)\,\mathrm{d}v$. Using this integral in Proposition 3.1 we arrive at

$$\int_{M^n}\{(\mathrm{Ric}(\psi^T,\psi^T)-n^2(1+F)^2)+(\|B\|^2-n)\}\,\mathrm{d}v=0. \tag{4.1}$$

Using the Schwarz inequality together with Lemma 2.2(i), we get $\|B\|^2 \geqslant nF^2$. Thus we have

$$0 \leqslant n \int_{M^n} (1+F)^2 \, \mathrm{d}v = \int_{M^n} n(F^2 - 1) \, \mathrm{d}v \leqslant \int_{M^n} (\|B\|^2 - n) \, \mathrm{d}v. \tag{4.2}$$

These two equations together with the hypothesis imply that $\int_{M^n} (\|B\|^2 - n) \, \mathrm{d}v = 0$, which in view of (4.2) gives that $1 + F = 0$. Hence by Theorem 4.1 we get that $\psi(M^n) \subset S^{n+p-1}(c)$ for some constant $c > 0$.

THEOREM 4.2 *Let $\psi : M^n \to \mathbb{R}^{n+p}$ be an oriented compact and connected submanifold with parallel mean curvature vector field. Then the condition $h(X, \psi^T) = 0$, $X \in \mathfrak{X}(M^n)$, holds if and only if $\psi(M^n) \subset S^{n+p-1}(c)$ for some constant $c > 0$.*

Proof. If $\psi(M^n) \subset S^{n+p-1}(c)$ for some constant $c > 0$, then by Theorem 4.1 F is a constant, and $\psi^T = 0$. Thus the given condition is satisfied. Conversely, using Propositions 3.1 and 3.2 we get

$$\int_{M^n} (\|B\|^2 - n) \, \mathrm{d}v = \int_{M^n} \left(\sum_i \|h(e_i, \psi^T)\|^2 \right) \mathrm{d}v.$$

Thus the hypothesis gives $\int_{M^n} (\|B\|^2 - n) \, \mathrm{d}v = 0$. This equation together with (4.2) proves that $F = -1$, and consequently that $\psi(M^n) \subset S^{n+p-1}(c)$ for some constant $c > 0$.

THEOREM 4.3 *Let $\psi : M^n \to \mathbb{R}^{n+p}$ be an oriented compact and connected submanifold. Then $\psi(M^n) \subset S^{n+p-1}(c)$ for some constant $c > 0$, if and only if the vector field ψ^T is harmonic.*

Proof. Suppose that ψ^T is harmonic, that is,

$$\Delta \psi^T = \sum_i [\nabla_{e_i} \nabla_{e_i} \psi^T - \nabla_{\nabla_{e_i} e_i} \psi^T] = 0$$

for a local orthonormal frame $\{e_1, \ldots, e_n\}$. Since $BX = -X + \nabla_X \psi^T$, $X \in \mathfrak{X}(M^n)$ and $(\nabla B)(X, Y) = \nabla_X \nabla_Y \psi^T - \nabla_{\nabla_X Y} \psi^T$ we get

$$\sum_i (\nabla B)(e_i, e_i) = \Delta \psi^T = 0.$$

As ψ^T is harmonic the above equation together with Lemma 2.2 gives $n \operatorname{grad} F = -Q(\psi^T)$. Taking the inner product with ψ^T in this last equation we arrive at

$$\operatorname{Ric}(\psi^T, \psi^T) = -n\xi(F) = -n \left\{ \operatorname{div}(F\psi^T) - F \operatorname{div} \psi^T \right\},$$

which gives

$$\operatorname{Ric}(\psi^T, \psi^T) + \operatorname{div}(nF\psi^T) - n^2 F(1+F) = 0.$$

Integrating this last equation and using Lemma 2.1 and Proposition 3.1, we get

$$\int_{M^n} (\|B\|^2 - n) \, \mathrm{d}v = 0.$$

Then an argument similar to that in equation (4.2) yields $F = -1$, and consequently this together with Theorem 4.1 implies that M^n lies on a sphere. The converse is clear.

As an immediate easy consequence of above theorem we have the following.

COROLLARY 4.2 *Let $\psi : M^n \to \mathbb{R}^{n+p}$ be a compact and connected submanifold. Then $\psi(M^n) \subset S^{n+p-1}(c)$ for some constant $c > 0$, if and only if the vector field ψ^T is killing.*

5. Submanifolds with parallel mean curvature vector

Let $\psi : M^n \to \mathbb{R}^{n+2}$ be an immersion of a compact manifold with parallel mean curvature vector field, that is, the mean curvature vector field H satisfies $\nabla^{\perp}_X H = 0$, $X \in \mathfrak{X}(M^n)$. Then $\|H\| = \alpha$, a non-zero constant. We choose a local orthonormal frame $\{N_1, N_2\}$ of normals such that $H = \alpha N_1$. Then in this case we have

$$\psi^{\perp} = \frac{F}{\alpha} N_1 + f N_2, \quad f = \langle \psi^{\perp}, N_2 \rangle. \tag{5.1}$$

Since $\nabla^{\perp}_X H = 0$, for each $X \in \mathfrak{X}(M^n)$, it follows that $\nabla^{\perp}_X N_1 = 0$, $X \in \mathfrak{X}(M^n)$, and as the codimension is 2 and $N_1 \perp N_2$, we also get that $\nabla^{\perp}_X N_2 = 0$, $X \in \mathfrak{X}(M^n)$. Define $C(X) = A_{N_2}(X)$, $X \in \mathfrak{X}(M^n)$, then it can be easily shown that C satisfies

$$\operatorname{tr} C = 0, \quad (\nabla C)(X, Y) = (\nabla C)(Y, X), \quad \sum_i (\nabla C)(e_i, e_i) = 0 \tag{5.2}$$

for $X, Y \in \mathfrak{X}(M^n)$ and a local orthonormal frame $\{e_1, \ldots, e_n\}$.

THEOREM 5.1 *Let (M^n, g) be oriented, positively curved, compact and connected. If there exists a positive constant λ such that the scalar curvature S of M^n satisfies $S < n(n-1)\lambda^{-2}$, then no isometric immersion of M^n in $\mathbb{R}^{n+2}$ with parallel mean curvature vector field is contained in a ball of radius λ.*

Proof. Suppose that there is an isometric immersion $\psi : M^n \to \mathbb{R}^{n+2}$ with parallel mean curvature vector and $\|\psi\| \leqslant \lambda$. Define $\varphi : M^n \to R$ by $\varphi = \frac{1}{2}\|C\|^2$, where C is as above. The Laplacian $\Delta\varphi$ is given by

$$\Delta\varphi = \|\nabla C\|^2 + \frac{1}{2} \sum_{i<j} (\mu_i - \mu_j)^2 K_{ij},$$

where μ_i are the eigenvalues of C. Integrating the above equation and using $K_{ij} > 0$, we get $\nabla C = 0$. This proves that $C = \mu I$ which, together with $\operatorname{tr} C = 0$, gives $C = A_{N_2} = 0$.

Now from (5.1), for any $X \in \mathfrak{X}(M^n)$, we get

$$X(f) = X\langle \psi^{\perp}, N_2 \rangle = -\langle h(X, \psi^T), N_2 \rangle = 0;$$

consequently f is a constant, call it β. Thus

$$\|\psi\|^2 = \|\psi^T\|^2 + \frac{F^2}{\alpha^2} + \beta^2 \tag{5.3}$$

and $B = A_{\psi^{\perp}} = (F/\alpha)A_{N_1}$, which gives $\|B\|^2 = (F^2/\alpha^2)\|A_{N_1}\|^2$. Using this in Proposition 3.1, we arrive at

$$\int_{M^n} \left\{ \operatorname{Ric}(\psi^T, \psi^T) + \frac{F^2}{\alpha^2}(\|A_{N_1}\|^2 - n^2\alpha^2) + n(n-1) \right\} \mathrm{d}v = 0.$$

The length $\|h\|^2$ of the second fundamental form is given by $\|h\|^2 = \sum_{\gamma=1}^{2} \|A_{N_\gamma}\|^2$ and $A_{N_2} = 0$. Thus we have $\|A_{N_1}\|^2 = \|h\|^2$, consequently the above integral takes the form

$$\int_{M^n} \left\{ \operatorname{Ric}(\psi^T, \psi^T) + \left(n(n-1) - \frac{F^2}{\alpha^2} S \right) \right\} \mathrm{d}v = 0,$$

where $S = n^2\|H\|^2 - \|h\|^2$ is the scalar curvature of M^n. Using equation (5.3) and $\|\psi\| \leqslant \lambda$ in above integral we get the inequality

$$\int_{M^n} \{(\mathrm{Ric}(\psi^T, \psi^T) + \|\psi^T\|^2 S + \beta^2 S) + \lambda^2(n(n-1)\lambda^{-2} - S)\}\,\mathrm{d}v \leqslant 0. \tag{5.4}$$

But the first bracket is non-negative and as $S < n(n-1)\lambda^{-2}$, the integrand is strictly positive and we get the contradiction.

As a consequence of above theorem we have the following.

COROLLARY 5.1 *Let M^n be an oriented positively curved compact and connected submanifold of $\mathbb{R}^{n+2}$ with parallel mean curvature vector. If M^n is contained in a ball of radius λ in $\mathbb{R}^{n+2}$ and the scalar curvature S of M^n satisfies $S \leqslant n(n-1)\lambda^{-2}$, then M^n lies on the hypersphere $S^{n+1}(\lambda^{-2})$ and indeed is isometric to $S^n(\lambda^{-2})$.*

Proof. The first term in the integral (5.4) is non-negative and so is the second term. Thus we get $S = n(n-1)\lambda^{-2}$ and $\mathrm{Ric}(\psi^T, \psi^T) + \|\psi^T\|^2 S + \beta^2 S = 0$, which consequently implies that $\psi^T = 0$ and $\beta = 0$. Then Theorem 4.1 with $\psi^T = 0$ implies that M^n lies on the hypersphere $S^{n+1}(\lambda^{-2})$. Note that the conclusion $\psi^T = 0$ implies that F is a constant, and by Lemma 2.1 the constant F must be -1. Also (5.3) gives that $\lambda^2 = 1/\alpha^2$. Substituting this in $S = n(n-1)\lambda^{-2}$, we get

$$S = n(n-1)\alpha^2. \tag{5.5}$$

Also the conclusion $\beta = 0$ gives that $\|h\|^2 = \|A_1\|^2$, where $A_1 = A_{N_1}$, $\{N_1, N_2\}$ being the local orthonormal frame chosen as in Theorem 5.1. We have $\mathrm{tr}\, A_1 = n\alpha$, and the Schwarz inequality gives $\|A_1\|^2 \geqslant 1/n(\mathrm{tr}\, A_1)^2 = n\alpha^2$ with equality holding if and only if $A_1 = \alpha I$. Thus using (2.8) we arrive at $S \leqslant n^2\alpha^2 - n\alpha^2 = n(n-1)\alpha^2$. Combining this last equation with (5.5) we get the equality in $\|A_1\|^2 \geqslant 1/n(\mathrm{tr}\, A_1)^2$, and consequently $A_1 = \alpha I$, that is, M^n is totally umbilical and hence isometric to a sphere of constant curvature $\alpha^2 = \lambda^{-2}$.

COROLLARY 5.2 *Let M^n be an oriented positively curved compact and connected submanifold of $\mathbb{R}^{n+2}$ with parallel mean curvature vector field H. If the scalar curvature S of M^n is constant, then M^n is $S^n(\alpha^2)$, where $\alpha = \|H\|$.*

COROLLARY 5.3 *Let M^n be an oriented compact, connected and positively curved submanifold of $\mathbb{R}^{n+2}$ with parallel mean curvature vector field. If $\mathrm{Ric}(\psi^T, \psi^T) \geqslant n^2(1+F)^2$, then M^n is an n-sphere $S^n(c)$.*

The proofs of these are similar to that of Corollary 5.1.

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