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Spherical submanifolds in a Euclidean space*

By

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Abstract. In this paper, we are interested in extending the study of spherical curves in R^3 to the submanifolds in the Euclidean space R^{n+p}. More precisely, we are interested in obtaining conditions under which an n-dimensional compact submanifold M of a Euclidean space R^{n+p} lies on the hypersphere $S^{n+p-1}(c)$ (standardly imbedded sphere in R^{n+p} of constant curvature c). As a by-product we also get an estimate on the first nonzero eigenvalue of the Laplacian operator Δ of the submanifold (cf. Theorem 3.5) as well as a characterization for an n-dimensional sphere $S^n(c)$ (cf. Theorem 4.1).

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1. Introduction

Given an n-dimensional submanifold M of a Euclidean space R^{n+p} with immersion $\psi : M \to R^{n+p}$, the position vector field ψ plays an important role in studying the geometry of the submanifolds (cf. [3], [8], [13], [14]). In particular, for compact hypersurfaces it is used to obtain a very useful tool, the Minkowski's formula, and this formula has proved to be very useful in studying the geometry of hypersurfaces (cf. [4]–[7], [9], [10], [15], [16]). The Minkowski's formula, is generalized to submanifolds in Euclidean space (cf. [2], Lemma 2.1). One of the interesting questions in the theory of curves in R^3 is to characterize spherical curves, that is to obtain conditions under which a unit speed curve lies on a sphere $S^2(c) \subset R^3$. There are several important characterizations for spherical curves (cf. [18], [19]). In [2], the question of extending the study of spherical curves in R^3 to submanifolds in Euclidean space is considered, namely to obtain conditions under which a compact submanifold $\psi : M \to R^{n+p}$ satisfies $\psi(M) \subset S^{n+p-1}(c)$, where $S^{n+p-1}(c)$ is the hypersphere of constant sectional curvature c in the Euclidean space R^{n+p}. In this paper we continue this study of obtaining conditions under which a compact submanifold of Euclidean space lies on a hypersphere. We call submanifolds lying

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on a hypersphere as a spherical submanifold in analogy to spherical curves. In Section 3, first we obtain two conditions characterizing spherical submanifolds (cf. Theorems 3.1 and 3.2) and then obtain a condition under which the normal component $\psi^{\perp}$ of the position vector field is umbilical section and use it to prove that for umbilic section $\psi^{\perp}$ on a compact Einstein submanifold M either $\psi(M) \subset S^{n+p-1}(c)$ or else M is isometric to a sphere (cf. Theorem 3.4). In this section we also prove that a compact submanifold M of R^{n+p} with constant scalar curvature S and umbilical normal section $\psi^{\perp}$ either lies on a hypersphere $S^{n+p-1}(c)$ or else the first nonzero eigenvalue λ_1 of the Laplacian operator Δ on M satisfies $S \geqslant (n-1)\lambda_1$ (cf. Theorem 3.5). As a corollary, this result proves that a submanifold with constant scalar curvature S and umbilical normal section $\psi^{\perp}$ satisfying $S < (n-1)\lambda_1$ necessarily lies on a hypersphere $S^{n+p-1}(c)$.

Let $\psi : M \to R^{n+p}$ be an n-dimensional compact submanifold with mean curvature vector field H. Then if we write $\psi = \psi^T + \psi^{\perp}$, $\psi^T, \psi^{\perp}$ being the tangential and normal components of ψ restricted to M, we have the smooth function $F = \langle H, \psi^{\perp}\rangle$, where $\langle , \rangle$ is the Euclidean metric on R^{n+p}. There are two operators A, B on the submanifold naturally associated to the position vector field ψ, defined by $A = A_{\psi^{\perp}}$, the Weingarten map with respect to the normal section $\psi^{\perp}$ and B is the symmetric operator associated to the Hessian H_F of the smooth function F by $H_F(X,Y) = g(BX,Y)$, $X, Y \in \mathfrak{X}(M)$, where g is the induced metric and $\mathfrak{X}(M)$ is the Lie-algebra of smooth vector fields on M. We use the operators A and B to define the number $\mu(M)$ for the submanifold $\psi : M \to R^{n+p}$ by

$$\mu(M) = \int_M \left((n-1)\|B + cA + cI\|^2 - 2nc\left(\mathrm{tr}AB + \|\nabla F\|^2\right)\right)dV,$$

where $c = \inf \frac{1}{n-1}\mathrm{Ric}$, Ric being the Ricci curvature of the submanifold and ∇F is the gradient of the smooth function F. This number $\mu(M)$ we use in Section 4 to obtain a characterization of a sphere $S^n(c)$ in R^{n+p} (cf. Theorem 4.1). The motivation for this number comes from the following observation: consider the n-dimensional sphere $S^n(c)$ of constant curvature c and $\psi : S^n(c) \to R^{n+p}$ the natural embedding then the second fundamental form of this submanifold satisfies $h(X,Y) = g(X,Y)H$, $\|H\|^2 = c$, $X, Y \in \mathfrak{X}(S^n(c))$. It is easy to see that $A = FI$, $\nabla F = -c\psi^T$ and $B = -c(1+F)I$ and that F satisfies $\Delta(1+F) = -nc(1+F)$ that is $1+F$ is an eigenfunction of the Laplacian operator Δ with first nonzero eigenvalue nc. Thus by the minimum principle one has $\int_{S^n(c)} \|\nabla F\|^2 dV = nc\int_{S^n(c)}(1+F)^2 dV$ and using $\int_{S^n(c)}(1+F)dV = 0$, one has $\int_{S^n(c)}(\mathrm{tr}AB + \|\nabla F\|^2)dV = 0$ and consequently that $\mu(S^n(c)) = 0$. This example raises the question "Does an n-dimensional compact submanifold of positive curvature in the Euclidean space R^{n+p} satisfying $\mu(M) = 0$ with nonconstant smooth function F necessarily a $S^n(c)$?" We answer this question in the affirmative and indeed prove a more general result (cf. Theorem 4.1) and Section 4 is devoted to the proof of this result.

2. Preliminaries

We denote by g and $\overline{\nabla}$ the Euclidean metric and the Euclidean connection on R^{n+p}. We also denote by the letter g and by ∇ the induced metric and the

Riemannian connection on the submanifold M. Then we have the following equations for the submanifold M

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.1}$$

$X, Y \in \mathfrak{X}(M)$, $N \in \Gamma(\upsilon)$, where $\mathfrak{X}(M)$ is the Lie-algebra of smooth vector fields, $\Gamma(\upsilon)$ is the space of smooth sections of the bundle υ of M, h the second fundamental form, A_N is the Weingarten map with respect to the normal $N \in \Gamma(\upsilon)$ which is related to the second fundamental form h by $g(A_N X, Y) = g(h(X, Y), N)$, $X, Y \in \mathfrak{X}(M)$ and $\nabla^{\perp}$ is the connection in the normal bundle υ. We also have the following equations of Gauss and Codazzi for the submanifold M

$$R(X, Y, Z, W) = g(h(Y, Z), h(X, W)) - g(h(X, Z), h(Y, W)) \tag{2.2}$$

$$(Dh)(X, Y, Z) = (Dh)(Y, Z, X) = (Dh)(Z, X, Y), \tag{2.3}$$

where R is the curvature tensor field of the submanifold M and

$$(Dh)(X, Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), X, Y, Z \in \mathfrak{X}(M).$$

The Ricci tensor Ric of the submanifold is given by

$$\mathrm{Ric}(X, Y) = ng(h(X, Y), H) - \sum_i g(h(X, e_i), h(Y, e_i)), \tag{2.4}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M and $H = \frac{1}{n}\sum_i h(e_i, e_i)$, is the mean curvature vector field. The Ricci operator Q is a symmetric operator defined by

$$\mathrm{Ric}(X, Y) = g(Q(X), Y), \ X, Y \in \mathfrak{X}(M).$$

The scalar curvature S of the submanifold M is given by

$$S = n^2\|H\|^2 - \|h\|^2, \tag{2.5}$$

where $\|h\|^2$ is the square of the length of the second fundamental form defined by $\|h\|^2 = \sum_{ij} \|h(e_i, e_j)\|^2$.

If we express $\psi = \psi^T + \psi^{\perp}$, where $\psi^T \in \mathfrak{X}(M)$ is the tangential component and $\psi^{\perp} \in \Gamma(\upsilon)$ is the normal component of ψ, and if we denote by $A = A_{\psi^{\perp}}$ the Weingarten map with respect to the normal vector field $\psi^{\perp} \in \Gamma(\upsilon)$, then using (2.1), we have

$$\nabla_X \psi^T = X + AX, \qquad \nabla^{\perp}_X \psi^{\perp} = -h(X, \psi^T), \quad X \in \mathfrak{X}(M). \tag{2.6}$$

We use the mean curvature vector field H to define a smooth function $F : M \to R$ on the submanifold M by $F = \langle H, \psi^{\perp} \rangle$. Then we have the following lemmas (cf. [2]) for an n-dimensional compact submanifold $\psi : M \to R^{n+p}$.

Lemma 2.1 ([2]). *Let M be an n-dimensional compact submanifold of the Euclidean space R^{n+p}. Then*

$$\int_M (1 + F) dv = 0.$$

Lemma 2.2 ([2]). *Let M be an n-dimensional submanifold of R^{n+p}. Then the tensor field A satisfies*

(i) $\operatorname{tr} A = nF$,
(ii) $(\nabla A)(X,Y) - (\nabla A)(Y,X) = R(X,Y)\psi^T$,
(iii) $\sum_i (\nabla A)(e_i, e_i) = n\nabla F + Q(\psi^T)$,

where $(\nabla A)(X,Y) = \nabla_X AY - A\nabla_X Y,\ X, Y \in \mathfrak{X}(M)$ *and* $\{e_1, \ldots, e_n\}$ *is a local orthonormal frame on M.*

Lemma 2.3 ([2]). *Let M be an n-dimensional compact submanifold of R^{n+p}. Then the tensor field A satisfies*

$$\int_M \left\{ \operatorname{Ric}(\psi^T, \psi^T) + \|A\|^2 - n^2F^2 + n(n-1) \right\} dV = 0.$$

Lemma 2.4. *Let M be an n-dimensional compact submanifold of the Euclidean space R^{n+p}. Then*

$$\int_M \left\{ \operatorname{Ric}(\nabla F, \psi^T) + n\|\nabla F\|^2 + \sum_i g(\nabla_{e_i}\nabla F, Ae_i) \right\} dV = 0,$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M.

Proof. Using Eqs. (2.4) and (2.6) we get

$$\begin{aligned}
\operatorname{Ric}(\nabla F, \psi^T) &= ng(H, h(\nabla F, \psi^T)) - \sum_i g(h(\nabla F, e_i), h(\psi^T, e_i)) \\
&= -ng\big(H, \nabla^\perp_{\nabla F}\psi^\perp\big) + \sum_i g(h(\nabla F, e_i), \nabla^\perp_{e_i}\psi^\perp) \\
&= -n\nabla F g(H, \psi^\perp) + ng(\nabla^\perp_{\nabla F}H, \psi^\perp) \\
&\quad + \sum_i \left[e_i g(h(\nabla F, e_i), \psi^\perp) - g(\nabla^\perp_{e_i} h\big(\nabla F, e_i), \psi^\perp\big) \right] \\
&= -n\|\nabla F\|^2 + ng(\nabla^\perp_{\nabla F}H, \psi^\perp) + \sum_i e_i g(A(\nabla F), e_i) \\
&\quad - \sum_i \left[g((Dh)(e_i, \nabla F, e_i), \psi^\perp) + g\big(h(\nabla_{e_i}\nabla F, e_i), \psi^\perp\big) \right] \\
&= -n\|\nabla F\|^2 + \operatorname{div}(A(\nabla F)) - \sum_i g(\nabla_{e_i}\nabla F, Ae_i),
\end{aligned}$$

where we have used the fact $\sum_i (Dh)(\nabla F, e_i, e_i) = n\nabla^\perp_{\nabla F}H$. Then the last equation proves the lemma.

For an n-dimensional compact Riemannian manifold (M, g) and the smooth function $F : M \to R$, the Hessian operator $B : \mathfrak{X}(M) \to \mathfrak{X}(M)$ corresponding to the Hessian H_F of the function F is defined by $H_F(X,Y) = g(BX, Y)$, $X, Y \in \mathfrak{X}(M)$. Then it is easy to see that the operator B has the following properties.

Lemma 2.5. *Let (M,g) be an n-dimensional compact Riemannian manifold. Then the operator B satisfies*

(i) $\mathrm{tr}B = \Delta F$,
(ii) $(\nabla B)(X,Y) - (\nabla B)(Y,X) = R(X,Y)\nabla F$,
(iii) $\sum_i (\nabla B)(e_i, e_i) = \nabla(\Delta F) + Q(\nabla F)$,

where $\Delta F = \mathrm{div}(\nabla F)$ *is the Laplacian of* F, $(\nabla B)(X,Y) = \nabla_X BY - B\nabla_X Y$, $X,Y \in \mathfrak{X}(M)$ *and* $\{e_1,\ldots,e_n\}$ *is a local orthonormal frame of* M.

Lemma 2.6. *The smooth function* $F : M \to R$ *on an n-dimensional compact Riemannian manifold* (M,g) *satisfies*

$$\int_M \{\mathrm{Ric}(\nabla F, \nabla F) + \|B\|^2 - (\Delta F)^2\} dV = 0.$$

Proof. Using (iii) of Lemma 2.5, we get

$$\mathrm{div}(B(\nabla F)) = \|B\|^2 + ng(\nabla F, \nabla(\Delta F)) + \mathrm{Ric}(\nabla F, \nabla F), \tag{2.7}$$

Also we have

$$\mathrm{div}(\Delta F(\nabla F)) = g(\nabla F, \nabla(\Delta F)) + (\Delta F)^2. \tag{2.8}$$

Using this equation in (2.7) and integrating we get the lemma.

3. Spherical submanifolds

Let $\psi : M \to R^{n+p}$ be an n-dimensional compact submanifold. Throughout this section by $S^{n+p-1}(c)$, we denote the sphere of constant curvature c naturally imbedded in the Euclidean space R^{n+p} as a hypersurface. First we have the following theorem.

Theorem 3.1. *Let* $\psi : M \to R^{n+p}$ *be an n-dimensional compact submanifold. Then a necessary and sufficient condition for* $\psi(M) \subset S^{n+p-1}(c)$ *is that* $\psi^T = 0$ *and* $F = -1$ *hold.*

Proof. Suppose $\psi(M) \subset S^{n+p-1}(c)$. Then $\|\psi\|^2 = \frac{1}{\sqrt{c}}$ holds. Using Eq. (2.6), we compute

$$\begin{aligned} \mathrm{div}\left(\frac{1}{2}\|\psi\|^2\psi^T\right) &= \|\psi^T\|^2 + \frac{n}{2}\|\psi\|^2(1+F) \\ &= \|\psi^T\|^2 + \frac{n}{2\sqrt{c}}(1+F). \end{aligned}$$

Which on integration together with Lemma 2.1 gives $\psi^T = 0$. Thus Eq. (2.6) confirms that $AX = -X$ and this together with (i) in Lemma 2.2 gives $F = -1$. Conversely, if $\psi^T = 0$ and $F = -1$, we get $X\left(\frac{1}{2}\|\psi\|^2\right) = 0$, $X \in \mathfrak{X}(M)$, which proves that $\|\psi\|^2 =$ a constant and consequently $\psi(M) \subset S^{n+p-1}(c)$ for some $c > 0$.

Remark. It is interesting to note that for $n = 1$ and $p = 2$, the Theorem 3.1 reduces to Wong's result in [18]. To see this let $\psi(s)$ be the unit speed curve in R^3. Then the unit tangent vector field T is given by $T = \overline{\nabla}_T\psi$ and in this case the mean curvature vector field H is given by $H = h(T,T)$. If $\kappa(s)$ is the curvature function, then $\overline{\nabla}_T T = \kappa N$ gives $H = kN$, where $\{N, B\}$ is orthonormal frame of normal vector fields. Now the conditions of Theorem 3.1, $\psi^T = 0$, $F = -1$ are equivalent to $\psi = \psi^\perp = -\rho N - fB$ and $\frac{1}{k} = \rho = -g(N, \psi)$, where $f = -g(B, \psi)$. Also as $\psi^T = 0$ from Eq. (2.6), we shall have $A_{\psi^\perp} = -I$, consequently $\rho A_N(T) + fA_B(T) = T$ which together with the covariant derivative with respect to T of the equation $\psi = -\rho N - fB$ gives $\rho' N + \rho g(\overline{\nabla}_T N, B)B + f'B + fg(\overline{\nabla}_T B, N)N = 0$. Equating N, B-components we get $\rho' = f\tau$ and $f' + \tau\rho = 0$, where $\tau = -g(\overline{\nabla}_T B, N)$, which are the equations of Wong characterizing the spherical curves in R^3 (cf. [18]).

Theorem 3.2. *Let $\psi : M \to R^{n+p}$ be an n-dimensional compact and connected submanifold. If $\psi(M)$ is contained in a ball of radius R and $H_0 = \sup \|H\|$, then $H_0 \geqslant R^{-1}$. The equality holds if and only if $\psi(M) \subset S^{n+p-1}\left(\frac{1}{R^2}\right)$ as a minimal submanifold.*

Proof. Since $|F| = |\langle H, \psi^\perp \rangle| \leqslant \|H\|\|\psi^\perp\| \leqslant \|H\|\|\psi\| \leqslant H_0 R$ which gives $1 - H_0 R \leqslant 1 + F \leqslant 1 + H_0 R$. Using Lemma 2.1, we get $H_0 \geqslant R^{-1}$. If $H_0 = R^{-1}$, we get $1 + F \geqslant 0$ which together with Lemma 2.1, gives $F = -1$. Also using

$$\operatorname{div}\left(\frac{1}{2}\|\psi\|^2 \psi^T\right) = \|\psi^T\|^2 + \frac{n}{2}\|\psi\|^2(1 + F),$$

we get $\psi^T = 0$, consequently by Theorem 3.1, we have $\psi(M) \subset S^{n+p-1}(c)$ for some $c > 0$. Moreover if H' is the mean curvature of M as submanifold in $S^{n+p-1}(c)$, then it follows that $H = H' - \sqrt{c}\overline{N}$, where $\overline{N} = \sqrt{c}\psi$ is the unit normal vector field to $S^{n+p-1}(c)$ in R^{n+p} (This follows from the fact that the second fundamental form h' of M in $S^{n+p-1}(c)$ is related to h by $h(X, Y) = h'(X, Y) - \sqrt{c}g(X, Y)\overline{N}$). Thus we have $\|H\|^2 = \|H'\|^2 + c$, which implies

$$R^{-2} = H_0^2 = \sup \|H\|^2 \geqslant \|H'\|^2 + c.$$

Also, $1 = H_0 R \geqslant \|H\|R \geqslant \|H\|\frac{1}{\sqrt{c}} \geqslant |g(H, \psi)| = |F| = 1$, implies $c = R^{-2}$. This together with above inequality gives $\|H'\| = 0$ and consequently $\psi : M \to S^{n+p-1}\left(\frac{1}{R^2}\right)$ is a minimal submanifold. Conversely if $\psi : M \to S^{n+p-1}\left(\frac{1}{R^2}\right)$ is a minimal submanifold, then the mean curvature field H of $\psi : M \to R^{n+p}$ is parallel. In particular $\|H\|^2 =$ constant, $H_0 = \|H\|$. Using this with $\psi^T = 0$ and $F = -1$ for $\psi(M) \subset S^{n+p-1}\left(\frac{1}{R^2}\right)$, we get $\psi = -\frac{1}{H_0^2}H$ which gives $H_0 = R^{-1}$.

Definition. Let $\psi : M \to R^{n+p}$ be a submanifold. A normal section $N \in \Gamma(\upsilon)$ is said to be an umbilical section if the Weingarten map A_N satisfies $A_N = fI$ for a smooth function $f : M \to R$.

We note that the Weingarten maps A_N in general do not satisfy Codazzi type of equation (cf. (ii) in Lemma 2.2) and therefore f need not be a constant. The following theorem gives condition under which $\psi^\perp$ is an umbilical section.

Theorem 3.3. *Suppose that $\psi : M \to R^{n+p}$ is an n-dimensional compact submanifold. If the Ricci tensor of M satisfies*

$$\operatorname{Ric}(\psi^T, \psi^T) \geqslant \frac{1}{2}(n-1)\|\psi^T\|^2 \Delta F,$$

then $\psi^\perp$ is an umbilical section.

Proof. For the smooth function $F = \langle H, \psi^{\perp} \rangle$, we have

$$\operatorname{div}\left(\frac{1}{2}\|\psi\|^2 \nabla F\right) = g(\nabla F, \psi^T) + \frac{1}{2}\|\psi\|^2 \Delta F. \tag{3.1}$$

Also

$$\operatorname{div}(F\psi^T) = g(\nabla F, \psi^T) + nF(1+F), \tag{3.2}$$

where we used $\operatorname{div}(\psi^T) = n(1+F)$ which is an outcome of Eq. (2.6). The Eqs. (3.1) and (3.2) together with Lemma 2.1 give

$$\frac{1}{2}\int_M (\|\psi\|^2 \Delta F)dV = n\int_M (F^2 - 1)dV, \tag{3.3}$$

using Eq. (3.3) in Lemma 2.3, we get

$$\int_M \left\{ \operatorname{Ric}(\psi^T, \psi^T) - \frac{1}{2}(n-1)\|\psi\|^2 \Delta F + (\|A\|^2 - nF^2) \right\} dV = 0.$$

Thus the statement together with the Schwarz's inequality $\|A\|^2 \geqslant nF^2$ gives

$$\operatorname{Ric}(\psi^T, \psi^T) = \frac{1}{2}(n-1)\|\psi\|^2 \Delta F \quad \text{and} \quad \|A\|^2 = nF^2,$$

with the second equality holds if and only if $A = FI$ which proves that $\psi^{\perp}$ is the umbilical section.

Remark. We note that in case of a unit speed curve ψ in R^3 as $h(T,T) = H$ we have $F = g(A_{\psi^{\perp}}T, T)$ and since $A_{\psi^{\perp}}(T) = g(A_{\psi^{\perp}}(T), T)T$ as the tangent space is 1-dimensional, we get $A_{\psi^{\perp}} = FI$, that is the normal section $\psi^{\perp}$ is an umbilic section. This is the motivation for studying the submanifolds in R^{n+p} with umbilic section $\psi^{\perp}$ for obtaining conditions on submanifolds to be spherical submanifolds.

Theorem 3.4. *Let $\psi : M \to R^{n+p}$ be an n-dimensional compact Einstein submanifold with umbilic section $\psi^{\perp}$. Then either $\psi(M) \subset S^{n+p-1}(c)$ or M is isometric to a sphere.*

Proof. We have $A = FI$, and consequently for a local orthonormal frame $\{e_1, \dots, e_n\}$, we get $\sum_i (\nabla A)(e_i, e_i) = \nabla F$. Moreover, M being an Einstein manifold, we get $Q(\psi^T) = \frac{S}{n}\psi^T$, where S is scalar curvature which is constant. Thus by (iii) in Lemma 2.3, we get $\nabla F = -\frac{S}{n(n-1)}\psi^T$. Taking $1 + F = \varphi$, we get $\nabla_X \nabla \varphi = -\frac{S}{n(n-1)}\nabla_X \psi^T = -\frac{S}{n(n-1)}\varphi X$, that is

$$H_{\varphi}(X,Y) = -\frac{S}{n(n-1)}\varphi g(X,Y), \quad X, Y \in \mathfrak{X}(M),$$

where H_{φ} is the Hessian of the smooth function φ. If φ is a constant, we get $F = -1$ and $\psi^T = 0$, that is by Theorem 3.1, $\psi(M) \subset S^{n+p-1}(c)$, if φ is not a constant, then $H_{\varphi} = -\frac{S}{n(n-1)}\varphi g$ implies M is isometric to an n-sphere by Obata's theorem (cf. [12]).

Theorem 3.5. *Let $\psi : M \to R^{n+p}$ be an n-dimensional compact submanifold with constant scalar curvature S. If the normal section $\psi^{\perp}$ is an umbilical section,* *then either $\psi(M) \subset S^{n+p-1}(c)$ or else the first nonzero eigenvalue λ_1 of the Laplacian operator Δ of M satisfies $S \geqslant (n-1)\lambda_1$.*

Proof. Since $A = FI$, using (*iii*) of Lemma 2.2, we get

$$\nabla F = -\frac{1}{n-1} Q(\psi^T). \tag{3.4}$$

Choosing a pointwise constant local orthonormal frame $\{e_1, \ldots, e_n\}$, we compute

$$\begin{aligned} \operatorname{div} Q(\psi^T) &= \sum_i e_i g(\psi^T, Q(e_i)) = \sum_i (1+F) g(e_i, Q(e_i)) \\ &\quad + \sum_i g(\psi^T, (\nabla Q)(e_i, e_i)), \\ &= (1+F)S + \frac{1}{2}\psi^T(S) \\ &= (1+F)S, \end{aligned}$$

where we have used the fact $\frac{1}{2}\nabla S = \sum_i (\nabla Q)(e_i, e_i)$ and that S is a constant. Thus for $\varphi = 1 + F$,

$$\Delta\varphi = \Delta F = -\frac{S}{(n-1)}\varphi. \tag{3.5}$$

If φ is a constant, then $F = -1$ and $\psi^T = 0$, and consequently by Theorem 3.1, $\psi(M) \subset S^{n+p-1}(c)$. If φ is not a constant, then by Eq. (3.5), we get $\lambda_1 \leqslant \frac{S}{(n-1)}$ and this proves the theorem.

As a direct consequence of the above theorem we have

Corollary 3.6. *Let $\psi : M \to R^{n+p}$ be an n-dimensional compact submanifold with constant scalar curvature S. If the normal section $\psi^\perp$ is an umbilical section, and the first nonzero eigenvalue λ_1 of the Laplacian operator Δ* of M *satisfies $S < (n-1)\lambda_1$, then $\psi(M) \subset S^{n+p-1}(c)$.*

Corollary 3.7. *Let $\psi : M \to R^{n+p}$ be an n-dimensional compact submanifold with constant scalar curvature S. If the normal section $\psi^\perp$ is an umbilical section, and M does not lie on a hypersphere, then first nonzero eigenvalue λ_1 of the Laplacian operator Δ of M satisfies $S \geqslant (n-1)\lambda_1$.*

We note that by the Schwarz inequality together with Eq. (2.5) we have $S \leqslant n(n-1)\|H\|^2$. Reilly [14] has shown that the first nonzero eigenvalue λ_1 of a compact submanifold satisfies $\lambda_1 \mathrm{Vol}(M) \leqslant n\int_M \|H\|^2 dV$. In Corollary 3.7, with additional restriction on M we got a sharper bound on λ_1.

4. A characterization of the sphere

In this section we consider an n-dimensional compact submanifold $\psi : M \to R^{n+p}$, with $c = \inf \frac{1}{(n-1)}\mathrm{Ric}$, and the number $\mu(M)$ defined by

$$\mu(M) = \int_M \big((n-1)\|B + cA + cI\|^2 - 2nc(\mathrm{tr}AB + \|\nabla F\|^2)\big) dV,$$

and prove the following characterization for the spheres.

Theorem 4.1. *Let $\psi : M \to R^{n+p}$ be an n-dimensional compact and connected submanifold with* $\inf \frac{1}{(n-1)}\mathrm{Ric} = c > 0$. *If the function F is not a constant and* $\mu(M) = 0$, *then M is isometric to* $S^n(c)$, *the n-sphere of constant curvature c in the Euclidean space* R^{n+1}.

Proof. We note that the operators A, B and $B + cB$ are symmetric with $\mathrm{tr}AB = \mathrm{tr}BA$ holds, and consequently

$$\|B + cA + cI\|^2 = \|B\|^2 + c^2\|A\|^2 + nc^2 + 2c\mathrm{tr}AB + 2nc^2F + 2c\Delta F \qquad (4.1)$$

where we used $\mathrm{tr}B = \Delta F$ and $\mathrm{tr}A = nF$ (cf. Lemma 2.2 and Lemma 2.5)

$$\mathrm{Ric}(\nabla F + c\psi^T, \nabla F + c\psi^T) = \mathrm{Ric}(\nabla F, \nabla F) + c^2\mathrm{Ric}(\psi^T, \psi^T) + 2c\mathrm{Ric}\big(\nabla F, \psi^T\big).$$

which together with Lemmas 2.3, 2.4 and 2.6 give

$$\int_M \{\{\mathrm{Ric}\big(\nabla F + c\psi^T, \nabla F + c\psi^T\big)\}\}dV$$
$$= \int_M \big\{ -\|B\|^2 + (\Delta F)^2 - c^2\|A\|^2 + n^2c^2F^2$$
$$- n(n-1)c^2 - 2nc\|\nabla F\|^2 - 2c\mathrm{tr}AB\big\}dV.$$

Using $\|B\|^2 \geqslant \frac{1}{n}(\Delta F)^2$ and $\|A\|^2 \geqslant nF^2$ in above equation we get

$$\int_M \big\{\mathrm{Ric}\big(\nabla F + c\psi^T, \nabla F + c\psi^T\big)\big\}dV$$
$$\leqslant \int_M \big\{(n-1)\big[\|B\|^2 + c^2\|A\|^2 - nc^2\big]$$
$$- 2nc\|\nabla F\|^2 - 2c\mathrm{tr}AB\big\}dV.$$

Now using Eq. (4.1), together with Lemma 2.1, and Stokes' theorem, we arrive at

$$\int_M \big\{\mathrm{Ric}\big(\nabla F + c\psi^T, \nabla F + c\psi^T\big)\big\}dV$$
$$\leqslant \int_M \big((n-1)\|B + cA + cI\|^2 - 2nc\big(\mathrm{tr}AB + \|\nabla F\|^2\big)\big)dV = \mu(M).$$

Since M is positively curved, using the statement $\mu(M) = 0$, we get

$$\nabla F = -c\psi^T. \qquad (4.2)$$

Note that Eq. (4.2) implies $B + cA + cI = 0$ and consequently the condition $\mu(M) = 0$ reduces to

$$\int_M (\mathrm{tr}AB + \|\nabla F\|^2)dV = 0. \qquad (4.3)$$

Using Eqs. (4.2) and (4.3) in Lemma 2.4, we arrive at

$$\int_M \big(\mathrm{Ric}(\nabla F, \nabla F) - (n-1)c\|\nabla F\|^2\big)dV = 0. \qquad (4.4)$$

However as $c = \inf \frac{1}{(n-1)}\mathrm{Ric}$, $\mathrm{Ric}(\nabla F, \nabla F) \geqslant (n-1)c\|\nabla F\|^2$, Eq. (4.4) gives

$$\mathrm{Ric}(\nabla F, \nabla F) = (n-1)c\|\nabla F\|^2.$$

This together with Eq. (4.2) gives

$$\mathrm{Ric}(\psi^T, \psi^T) = (n-1)c^{-1}\|\nabla F\|^2$$

and using this in Lemma 2.3, we arrive at

$$\int_M \left((n-1)\|\nabla F\|^2 - n(n-1)c(1+F)^2 + c(\|A\|^2 - nF^2\right)dV = 0, \tag{4.5}$$

where we have used the fact $\int_M (F^2 - 1)dV = \int_M (1+F)^2 dV$, which is an outcome of Lemma 2.1. Also Lemma 2.1 together with the minimum principle implies

$$\int_M \|\nabla F\|^2 dV \geqslant \lambda_1 \int_M (1+F)^2 dV,$$

where λ_1 is the first nonzero eigenvalue of the Laplacian operator. Thus Eq. (4.5) takes the form

$$(n-1)(\lambda_1 - nc)\int_M (1+F)^2 dV + c\int_M \left(\|A\|^2 - nF^2\right)dV \leqslant 0. \tag{4.6}$$

Moreover, it is known that if $\mathrm{Ric} \geqslant (n-1)c$, then $\lambda_1 \geqslant nc$ and equality holds if and only if $M = S^n(c)$. Thus by inequality (4.6), we conclude that $M = S^n(c)$.

It is interesting to note that for the special case $p = 1$ in Theorem 4.1, we can drop the condition that F is not a constant, as for the case $\psi : M \to R^{n+1}$, F is a constant implies $\psi = \psi^\perp$ and consequently $A = -I$ and thus the second fundamental form takes the form $h(X, Y) = -\frac{1}{\|\psi\|} g(X, Y)N$, where N is the unit normal vector field. Consequently $M = S^n(c)$, where $\|\psi\|^2 = \frac{1}{c}$. Thus we have

Corollary. *Let $\psi : M \to R^{n+1}$ be an n-dimensional compact hypersurface with* $\inf \frac{1}{(n-1)}\mathrm{Ric} = c > 0$. *If $\mu(M) = 0$, then $M = S^n(c)$.*

References

[1] Alexandrov AD (1962) A characteristic property of spheres. Ann Mat Pure Appl **58**: 303–315
[2] Alodan H, Deshmukh S (2002) Spherical submanifolds of a Euclidean space. Quart J Math **53**: 249–256
[3] Chen BY (1984) Total Mean Curvature and Submanifolds of Finite type. Singapore: World Scientific
[4] Chern SS (1959) Integral formulas for hypersurfaces in Euclidean space and their applications to uniqueness theorems. J Math Mech **8**: 947–955
[5] Deshmukh S (1992) Isometric immersion of a compact Riemannian manifold into a Euclidean space. Bull Aust Math Soc **46**: 177–178
[6] Deshmukh S (1992) An integral formula for compact hypersurfaces in a Euclidean space and its applications. Glasgow Math J **34**: 309–311
[7] Garay OJ (1989) An application of Reilly's formula. Bull Lond Math Soc **21**: 176–178
[8] Gardner RB (1977) New viewpoints in the geometry of submanifolds of R^N. Bull Amer Math Soc **83**: 1–35
[9] Halpern B (1971) On the immersion of an n-dimensional manifold in n + 1-dimensional Euclidean space. Proc Amer Math Soc **30**: 181–184
[10] Hsiung CC (1951) Some integral formulas for closed hypersurfaces. Math Scand **2**: 286–294

[11] Montiel S, Ros A (1991) Compact hypersurfaces: the Alexandrov theorem for higher order mean curvature. In: Lawson HB et al (eds) Differential Geometry, pp 279–296, Harlow: Pitman
[12] Obata M (1962) Certain conditions for a Riemannian manifold to be isometric with a sphere. J Math Soc Japan **14**: 333–340
[13] Pigola S, Rigoli M, Setti AG (2003) Some applications of integral formulas in Riemannian geometry and PDE's. Milan J Math **71**: 219–281
[14] Reilly R (1977) On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space. Comment Math Helv **52**: 525–533
[15] Rigoli M (1988) On immersed compact submanifolds of Euclidean space. Proc Amer Math Soc **102**: 153–156
[16] Ros A (1988) Compact hypersurfaces with constant scalar curvature and a congruence theorem. J Diff Geom **27**: 215–223
[17] Ros A (1987) Compact hypersurfaces with constant higher order mean curvatures. Rev Mat Iberoamericana **3**: 447–453
[18] Wong Y (1963) A global formulation of the condition for a curve to lie in a sphere. Monats Math **67**: 363–365
[19] Wong Y (1972) On an explicit characterization of spherical curves. Proc Amer Math Soc **34**: 239–242

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