The Journal of Geometric Analysis
Volume 15, Number 4, 2005

Curvature Bounds for the Spectrum of a Compact Riemannian Manifold of Constant Scalar Curvature

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ABSTRACT. Let (M, g) be an n-dimensional compact and connected Riemannian manifold of constant scalar curvature. If the sectional curvatures of M are bounded below by a constant $\alpha > 0$, and the Ricci curvature satisfies $\operatorname{Ric} \leq (n-1)\alpha\delta$, $\delta \geq 1$, *then it is shown that either M is isometric to the n-sphere $S^n(\alpha)$ or else each nonzero eigenvalue λ of the Laplacian acting on the smooth functions of M satisfies the following:*

$$\lambda^2 + 3n\alpha(\delta - 2)\lambda + 2n\alpha^2\delta(1 + (n-1)\delta) > 0 .$$

1. Introduction

Reading the spectrum of the Laplacian operator acting on the smooth functions of a Riemannian manifold using its geometric data is one of the fascinating fields in the Geometry of Riemannian manifolds. In particular, the study of obtaining lower bounds for the eigenvalues of the Laplacian operator Δ acting on the smooth functions on a Riemannian manifold (M, g) using its curvature information was first initiated by Lichnerowicz (cf. Theorem 4.70 p. 210 in [3]). This result states that if M is an n-dimensional compact Riemannian manifold whose Ricci curvature satisfies $\operatorname{Ric} \geq (n-1)k$, $k > 0$, then the first eigenvalue λ_1 of Δ satisfies $\lambda_1 \geq nk$. For closed Einstein manifolds of dimension n, Simon [7] has shown that either they are isometric to n-sphere or else each eigenvalue of their Laplacian operators satisfies $\lambda > 2nk_0$, where k_0 is the infimum of the sectional curvatures. Similar results are obtained by Tanno [8] for Einstein manifolds, and by the first author [2] for Einstein-like manifolds.

The class of compact Riemannian manifolds with constant scalar curvature properly contains Einstein manifolds as well as Einstein-like manifolds, as well as the fact that the metric on a compact Riemannian manifold (dimension greater than two) can be conformally deformed to a metric of constant scalar curvature [6, 9], makes it of natural interest to find bounds on the eingenvalues of the Laplacian operator for compact Riemannian manifolds of constant scalar curvature.

Math Subject Classifications. Primary: 53C20, 58G25; secondary 53B25, 53C20, 53C40.
Key Words and Phrases. Scalar curvature, lower bounds of eigenvalues, isometric to a sphere.

ISSN 1050-6926

In this article we are interested in obtaining bounds for the eigenvalues of the Laplacian operator for a compact Riemannian manifold (M, g) of constant scalar curvature whose sectional curvatures are bounded below by a constant $\alpha > 0$, and the Ricci curvature satisfies $\mathrm{Ric} \leq (n-1)\alpha\delta$, $\delta \geq 1$. The main result of this article is the following:

Theorem. *Let (M, g) be an n-dimensional compact and connected Riemannian manifold of constant scalar curvature. If the sectional curvatures of M are bounded below by a constant $\alpha > 0$, and the Ricci curvature satisfies $\mathrm{Ric} \leq (n-1)\alpha\delta$, $\delta \geq 1$, then either M is isometric to the n-sphere $S^n(\alpha)$ or else each nonzero eigenvalue λ of the Laplacian satisfies following:*

$$\lambda^2 + 3n\alpha(\delta - 2)\lambda + 2n\alpha^2\delta(1 + (n-1)\delta) > 0 .$$

Note that sectional curvatures of a compact Riemannian manifold lie in a closed interval of the Real line and so are the Ricci curvatures. Thus, the constraint on the curvatures in the above theorem is only on sectional curvatures is being required to be positive. We also get the result of Simon [7], as a particular case of our result (cf. Corollary 4.1).

As a particular case of above theorem we also have the following corollary:

Corollary. Let (M, g) be a compact and connected Riemannian manifold of constant curvature $c > 0$. Then either M is isometric to the n-sphere $S^n(c)$ or else each nonzero eigenvalue of the Laplacian satisfies $\lambda > 2nc$.

Some characterizations for spheres are also obtained using the eigenvalues of the Laplacian, bounds on the Ricci curvature and a certain function defined on the Riemannian manifold of constant scalar curvature (cf. Theorems 5.1, 5.2).

2. Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold, with Riemannian connection ∇. Throughout this article the Riemannian manifolds are of class C^∞ and without boundary. We denote by $C^\infty(M)$ the ring of smooth functions on M and by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields on M. The curvature tensor field R and the Ricci tensor field Ric of M are given by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad \mathrm{Ric}(X, Y) = \sum_{i=1}^{n} R(e_i, X; Y, e_i) ,$$

$X, Y, Z \in \mathfrak{X}(M)$, where $R(X, Y; Z, W) = g(R(X, Y)Z, W)$, and $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M.

The Ricci operator $Q : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by $g(Q(X), Y) = \mathrm{Ric}(X, Y)$ and the divergence F of the curvature tensor field R is a tensor field defined by

$$F(X, Y) = \sum_{i=1}^{n}(\nabla_{e_i} R)(X, Y)e_i, \quad X, Y \in \mathfrak{X}(M) .$$

Using the curvature properties and the Bianchi identities, it is easy to verify that F satisfies

$$F(X, Y) = -F(Y, X), F(X, fY) = fF(X, Y) = F(fX, Y), \quad f \in C^\infty(M) , \tag{2.1}$$

and that

$$g(F(X, Y), Z) + g(F(Y, Z), X) + g(F(Z, X), Y) = 0, \quad X, Y, Z \in \mathfrak{X}(M) . \tag{2.2}$$

From Equations (2.1), (2.2), and the second Bianchi identity we immediately obtain the following.

Lemma 2.1. $(\nabla Q)(X, Y) = (\nabla Q)(Y, X) - F(X, Y), \quad X, Y \in \mathfrak{X}(M),$
where $(\nabla Q)(X, Y) = \nabla_X Q(Y) - Q(\nabla_X Y)$.

Lemma 2.2. *The gradient of the scalar curvature S satisfies*

$$\frac{1}{2}\,\text{grad}\ S = \sum_j (\nabla Q)(e_j, e_j)\,.$$

The Hessian H_f of $f \in C^\infty(M)$ is given by $H_f(X, Y) = g(\nabla_X\ \text{grad}f, Y)$, and using Lemma 2.2 it can be easily shown that the Hessian H_s of the scalar curvature S satisfies

$$\frac{1}{2}H_s(X, Y) = \sum_i g\big((\nabla^2 Q)(X, e_i, e_i), Y\big), \quad X, Y \in \mathfrak{X}(M)\,, \tag{2.3}$$

where the second covariant derivative $(\nabla^2 Q)(X, Y, Z)$ is given by

$$\big(\nabla^2 Q)(X, Y, Z\big) = \nabla_X(\nabla Q)(Y, Z) - (\nabla Q)(\nabla_X Y, Z) - (\nabla Q)(Y, \nabla_X Z)\,.$$

For $f \in C^\infty(M)$, we define an operator $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $A(X) = \nabla_X$ gradf. This operator has the following properties

$$\text{tr}\,.A = -\Delta f, \quad g(AX, Y) = g(X, AY), \quad X, Y \in \mathfrak{X}(M)\,, \tag{2.4}$$

where Δ is the Laplacian operator defined by $\Delta\, f = -\,\text{div}(\text{gradf}),\ f \in C^\infty(M)$. The covariant derivative (∇A) of the operator A satisfies

$$(\nabla A)(X, Y) = (\nabla A)(Y, X) + R(X, Y)\ \text{gradf}\ , \tag{2.5}$$

$$g((\nabla A)(X, Y), Z) = g(Y, (\nabla A)(X, Z)), X, Y \in \mathfrak{X}(M)\,, \tag{2.6}$$

the first of which follows from the definition of A and the second from the fact that A is symmetric [see (2.4)]. Using the Ricci identity we get

$$\big(\nabla^2 A\big)(X, Y, Z) - \big(\nabla^2 A\big)(Y, X, Z) = R(X, Y)\, AZ - A(R(X, Y)Z)\,. \tag{2.7}$$

Lemma 2.3. *For a local orthonormal frame* $\{e_1, \ldots, e_n\}$ *on* M,

$$\sum_i (\nabla A)(e_i, e_i) = -\,\text{grad}\ \Delta f + Q(\text{gradf})\ ,$$

where $A(X) = \nabla_X$ gradf, $f \in C^\infty(M), \quad X \in \mathfrak{X}(M)$.

Proof. Using (2.4), $-\Delta f = \sum_i g(Ae_i, e_i)$, and thus for $X \in \mathfrak{X}(M)$ we have

$$\begin{aligned} -X(\Delta f) &= \sum_i g((\nabla A)(X, e_i), e_i) = \sum_i g((\nabla A)(e_i, X) + R(X, e_i)\,\text{gradf}, e_i) \\ &= \sum_i g(X, (\nabla A)(e_i, e_i)) + \sum_i R(X, e_i; \text{gradf}, e_i) \qquad (2.8) \\ &= \sum g((\nabla A)(e_i, e_i), X) - \text{Ric}(X, \text{gradf})\,, \end{aligned}$$

where we have used (2.2) and (2.3). The last equation proves the lemma. □

Finally, in this section, we compute the Hessian $H_{\Delta f}$ of the function Δf, $f \in C^\infty(M)$.

Lemma 2.4. *For $f \in C^\infty(M)$, the Hessian $H_{\Delta f}$ of the function Δf satisfies*

$$H_{\Delta f}(X, Y) = -\sum_i g\big((\nabla^2 A)(X, e_i, e_i), Y\big) + g\big(\nabla_X Q(\text{gradf}), Y\big), \quad X, Y \in \mathfrak{X}(M).$$

Proof. For $Y \in \mathfrak{X}(M)$, using Lemma 2.3, we compute

$$-Y(\Delta f) = \sum_i g(Y, (\nabla A)(e_i, e_i)) - g(Q(\text{gradf}), Y),$$

from which we arrive at

$$\begin{aligned} -H_{\Delta f}(X, Y) &= -XY(\Delta f) + \nabla_X Y(\Delta f) \\ &= \sum_i g\big(Y, (\nabla^2 A)(X, e_i, e_i)\big) - g(\nabla_X Q(\text{gradf}), Y), \end{aligned}$$

$X, Y \in \mathfrak{X}(M)$. This proves the lemma. □

3. Integral formulas

In this section we derive integral formulas which are the main ingredients in the proof of the main theorem. For $X \in \mathfrak{X}(M)$, we define the function $\|R_X\|^2$ called the square of the length of the curvature tensor field in the direction of X by

$$\|R_X\|^2 = \sum_{ij} g(R(e_i, e_j)X, R(e_i, e_j)X), \tag{3.1}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. For an n-dimensional Riemannian manifold (M, g) of constant curvature c we have $\|R_X\|^2 = 2(n-1)c^2 \|X\|^2$, $X \in \mathfrak{X}(M)$.

Proposition 3.1. *Let (M, g) be an n-dimensional compact Riemannian manifold without boundary and $f \in C^\infty(M)$. Then*

$$\int_M \Bigg\{ \|\nabla A\|^2 - \|\text{grad}\,\Delta f\|^2 + 2\,\text{Ric}(\text{gradf}, \text{grad}\,\Delta f) - \|Q(\text{gradf})\|^2 - \frac{1}{2}\|R_{\text{gradf}}\|^2 + \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij} \Bigg\}\, dv = 0$$

where $A(X) = \nabla_X$ gradf, and $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A with respect to a local orthonormal frame $\{e_1, \ldots, e_n\}$ and K_{ij} is the sectional curvature of the plane section spanned by $\{e_i, e_j\}$.

Proof. We compute

$$\begin{aligned}\operatorname{div}(A(Q(\mathrm{grad}f)) &= \sum_i g(\nabla_{e_i} AQ(\mathrm{grad}f), e_i) = \sum_i g((\nabla A)(e_i, Q(\mathrm{grad}f), e_i) \\ &\quad + \sum_i g(\nabla_{e_i} Q(\mathrm{grad}f), Ae_i) \\ &= \sum_i g(Q(\mathrm{grad}f), (\nabla A)(e_i, e_i)) + \sum_i g(\nabla_{e_i} Q(\mathrm{grad}f), Ae_i) \\ &= -\mathrm{Ric}(\mathrm{grad}f, \mathrm{grad}\ \Delta f) + \|Q(\mathrm{grad}f)\|^2 \\ &\quad + \sum_i g(\nabla_{e_i} AQ(\mathrm{grad}f), Ae_i)\end{aligned} \tag{3.2}$$

where we have used Lemma 2.3.

Similarly we have

$$\begin{aligned}\operatorname{div}(A(\mathrm{grad}\,\Delta f)) &= \sum_i g(\mathrm{grad}\,\Delta f, (\nabla A)(e_i, e_i)) + \sum_i g(\nabla_{e_i} \mathrm{grad}\,\Delta f, Ae_i) \\ &= -\|\mathrm{grad}\,\Delta f\|^2 + \mathrm{Ric}(\mathrm{grad}f, \mathrm{grad}\,\Delta f) \\ &\quad + \sum_i g(\nabla_{e_i} \mathrm{grad}\,\Delta f, Ae_i)\,.\end{aligned} \tag{3.3}$$

Next, consider the vector field $X = \sum_i (\nabla A)(e_i, Ae_i)$, where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. We have

$$\begin{aligned}\operatorname{div} X &= \sum_j g(\nabla_{e_j} X, e_j) = \sum_j e_j g(X, e_j) = \sum_{ij} e_j g((\nabla A)(e_i, Ae_i), e_j) \\ &= \sum_{ij} e_j g(Ae_i, (\nabla A)(e_i, e_j)) \\ &= \sum_{ij} g((\nabla A)(e_j, e_i), (\nabla A)(e_i, e_j)) + \sum_{ij} g\big(Ae_i, \big(\nabla^2 A\big)(e_j, e_i, e_j)\big)\,.\end{aligned} \tag{3.4}$$

Using Equation (2.7) and Lemma 2.4, we arrive at

$$\begin{aligned}\sum_{ij} g\big(Ae_i, \big(\nabla^2 A\big)(e_j, e_i, e_j)\big) &= \sum_{ij} g\big(Ae_i, \big(\big(\nabla^2 A\big)(e_i, e_j, e_j)\big) \\ &\quad + \sum_{ij} R(e_j, e_i; Ae_j, Ae_i) - \sum_{ij} R\big(e_j, e_i; e_j, A^2 e_i\big) \\ &= -\sum_i H_{\Delta f}(e_i; Ae_i) + \sum_i g(\nabla_{e_i} Q(\mathrm{grad}f), Ae_i) \\ &\quad + \sum_{ij} R(e_j, e_i; Ae_j, Ae_i) - \sum_{ij} R\big(e_j, e_i; e_j, A^2 e_i\big)\,.\end{aligned} \tag{3.5}$$

Using a local orthonormal frame $\{e_1, \ldots, e_n\}$ which diagonalizes A, that is, $A(e_i) = \lambda_i e_i$, we have

$$\sum_{ij} R(e_j, e_i; Ae_j, Ae_i) - \sum_{ij} R\big(e_j, e_i; e_j, A^2 e_i\big) = -\sum_{ij} \lambda_j \lambda_i K_{ij} + \sum_{ij} \lambda_i^2 K_{ij}$$

$$= \frac{1}{2}\left[2\sum_{ij} \lambda_i^2 K_{ij} - \sum_{ij} 2\lambda_j \lambda_i K_{ij}\right]$$

$$= \frac{1}{2}\left[\sum_{ij} \lambda_i^2 K_{ij} + \sum_{ij} \lambda_j^2 K_{ij} - \sum_{ij} 2\lambda_j \lambda_i K_{ij}\right]$$

$$= \frac{1}{2}\sum_{ij} (\lambda_i - \lambda_j)^2 K_{ij} = \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij}$$

where K_{ij} is the sectional curvature of the plane section spanned by $\{e_i, e_j\}$.

Thus, the Equation (3.5) takes the form

$$\sum_{ij} g\big(Ae_i, (\nabla^2 A)(e_j, e_i, e_j)\big) = -\sum_{i} g(\nabla_{e_i} \operatorname{grad} \Delta f, Ae_i) + \sum_{i} g(\nabla_{e_i} Q(\operatorname{gradf}), A(e_i)) + \sum_{i<j} (\lambda_j - \lambda_i)^2 K_{ij} \tag{3.6}$$

where we have used the definition of the Hessian $H_{\Delta f}$. Also, using Equation (2.5), we get

$$\begin{aligned} \|R_{\text{gradf}}\|^2 &= \sum_{ij} g(R(e_i, e_j)\operatorname{gradf}, R(e_i, e_j)\operatorname{gradf}) \\ &= \sum_{ij} g((\nabla A)(e_i, e_j) - (\nabla A)(e_j, e_i), (\nabla A)(e_i, e_j) - (\nabla A)(e_j, e_i)) \\ &= 2\,\|\nabla A\|^2 - 2\sum_{ij} g((\nabla A)(e_i, e_j), (\nabla A)(e_j, e_i)) \end{aligned}$$

consequently we arrive at

$$\sum_{ij} g((\nabla A)(e_i, e_j), (\nabla A)(e_j, e_i)) = \|\nabla A\|^2 - \frac{1}{2}\,\|R_{\text{gradf}}\|^2 .$$

Using this equation and (3.6) in (3.4), we get

$$\operatorname{div} X = \|\nabla A\|^2 - \frac{1}{2}\,\|R_{\text{gradf}}\|^2 - \sum_{i} g(\nabla_{e_i} \operatorname{grad} \Delta f, Ae_i) + \sum_{i} g(\nabla_{e_i} Q(\operatorname{gradf}), Ae_i) + \sum_{i<j} (\lambda_j - \lambda_i)^2 K_{ij} . \tag{3.7}$$

Finally, we use (3.2), (3.3), and (3.7) to arrive at

$$\operatorname{div} X + \operatorname{div}(A(\operatorname{grad} \Delta f)) - \operatorname{div}(A(Q(\operatorname{gradf}))) = \|\nabla A\|^2 - \|\operatorname{grad} \Delta f\|^2 + 2\operatorname{Ric}(\operatorname{gradf}, \operatorname{grad} \Delta f) - \|Q(\operatorname{gradf})\|^2 - \frac{1}{2}\,\|R_{\text{gradf}}\|^2 + \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij} .$$

Integrating this equation and using the Stokes theorem we get the desired result. □

Proposition 3.2. *Let (M, g) be an n-dimensional compact Riemannian manifold without boundary and $f \in C^{\infty}(M)$. Then*

$$\int_M \left\{ \sum_{ij} g((\nabla_{e_j} R)(e_j, e_i)\,\mathrm{grad} f, Ae_i) \right\} dv = \int_M \left\{ -\frac{1}{2} \|R_{\mathrm{grad} f}\|^2 + \sum_{ij} R(e_i, e_j; Ae_j, Ae_i) \right\} dv$$

where $A(X) = \nabla_X$ grad$f$.

Proof. Consider the vector field $X = \sum_i R(\mathrm{grad} f, Ae_i)e_i$ and compute

$$\begin{aligned}
\mathrm{div}\, X &= \sum_{ij} g(\nabla_{e_j} R(\mathrm{grad} f, Ae_i)e_i, e_j) \\
&= \sum_{ij} g((\nabla_{e_j} R)(\mathrm{grad} f, Ae_i)e_i, e_j) + \sum_{ij} g(R(Ae_j, Ae_i)e_i, e_j) \\
&\quad + \sum_{ij} g(R(\mathrm{grad} f, (\nabla A)(e_j, e_i))e_i, e_j) \\
&= \sum_{ij} (\nabla_{e_j} R)(\mathrm{grad} f, Ae_i; e_i, e_j) + \sum_{ij} R(Ae_j, Ae_i; e_i, e_j) \\
&\quad + \sum_{ij} R(\mathrm{grad} f, (\nabla A)(e_j, e_i); e_i, e_j) \\
&= -\sum_{ij} (\nabla_{e_j} R)(e_j, e_i; \mathrm{grad} f, Ae_i) + \sum_{ij} R(e_i, e_j; Ae_j, Ae_i) \\
&\quad + \sum_{ij} R(e_i, e_j; \mathrm{grad} f, (\nabla A)(e_j, e_i))\,.
\end{aligned}$$

Now using (2.5), we get

$$\begin{aligned}
\mathrm{div}\, X &= -\sum_{ij} g((\nabla_{e_j} R)(e_j, e_i)\,\mathrm{grad} f, Ae_i) + \sum_{ij} R(e_i, e_j; Ae_j, Ae_i) \\
&\quad + \sum_{ij} g((\nabla A)(e_i, e_j) - (\nabla A)(e_j, e_i), (\nabla A)(e_j, e_i)) \\
&= -\sum_{ij} g((\nabla_{e_j} R)(e_j, e_i)\,\mathrm{grad} f, Ae_i) + \sum_{ij} R(e_i, e_j; Ae_j, Ae_i) \\
&\quad + \sum_{ij} g((\nabla A)(e_i, e_j), (\nabla A)(e_j, e_i)) - \|\nabla A\|^2
\end{aligned}$$

finally, use $\frac{1}{2}\|R_{\mathrm{grad} f}\|^2 = \|\nabla A\|^2 - \sum_{ij} g((\nabla A)(e_i, e_j), (\nabla A)(e_j, e_i))$, which follows from Equation (2.5), in above equation and integrate to get the desired result. □

Proposition 3.3. *Let (M, g) be an n-dimensional compact Riemannian manifold of constant scalar curvature without boundary. Then for $f \in C^{\infty}(M)$,*

$$\int_M \left\{ 2\,\mathrm{Ric}(\mathrm{grad} f, \mathrm{grad}\, \Delta f) - \|Q(\mathrm{grad} f)\|^2 - 3 \sum_i \mathrm{Ric}(Ae_i, Ae_i) \right.$$

$$\left. - \sum_i R(e_i, \mathrm{grad} f; \mathrm{grad} f, Q(e_i)) + \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij} + \frac{1}{2} \|R_{\mathrm{grad} f}\|^2 \right\} dv = 0$$

where $A(X) = \nabla_X \operatorname{grad} f$, and $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A with respect to a local orthonormal frame $\{e_1, \ldots, e_n\}$ and K_{ij} is the sectional curvature of the plane section spanned by $\{e_i, e_j\}$.

Proof. We use Equations (2.3), (2.7), Lemma 2.1, and the Ricci identity for Q similar to that in (2.7) for A, to compute

$$
\begin{aligned}
\operatorname{div}((\nabla Q)(\operatorname{gradf}, \operatorname{gradf})) &= \sum_i g(\nabla_{e_i}(\nabla Q)(\operatorname{gradf}, \operatorname{gradf}), e_i) = \sum_i e_i g((\nabla Q)(\operatorname{gradf}, \operatorname{gradf}), e_i) \\
&= \sum_i g(Ae_i, (\nabla Q)(\operatorname{gradf}, e_i)) + \sum_i g(\operatorname{gradf}, \nabla_{e_i}(\nabla Q)(\operatorname{gradf}, e_i)) \\
&= \sum_i g(Ae_i, (\nabla Q)(e_i, \operatorname{gradf}) - F(\operatorname{gradf}, e_i)) \\
&\quad + \sum_i g\big(\operatorname{gradf}, \big(\nabla^2 Q\big)(e_i, \operatorname{gradf}, e_i) + (\nabla Q)(Ae_i, e_i)\big) \\
&= \sum_i g((\nabla Q)(e_i, Ae_i), \operatorname{gradf}) - \sum_i g(F(\operatorname{gradf}, e_i), Ae_i) \\
&\quad + \sum_i g((\nabla Q)(Ae_i, e_i), \operatorname{gradf}) + \sum_i g\big(\operatorname{gradf}, \big(\nabla^2 Q\big)(\operatorname{gradf}, e_i, e_i)\big) \\
&\quad + \sum_i R(e_i, \operatorname{gradf}; Q(e_i), \operatorname{gradf}) - \sum_i R(e_i, \operatorname{gradf}; e_i, Q(\operatorname{gradf})) .
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\operatorname{div}((\nabla Q)(\operatorname{gradf}, \operatorname{gradf})) &= 2\sum_i g((\nabla Q)(e_i, Ae_i), \operatorname{gradf}) - \sum_i g(F(\operatorname{gradf}, e_i)Ae_i) \\
&\quad + \|Q(\operatorname{gradf})\|^2 - \sum_i g(F(Ae_i, e_i)\operatorname{gradf}) \\
&\quad - \sum_i R(e_i, \operatorname{gradf}; \operatorname{gradf}, Q(e_i))
\end{aligned} \tag{3.8}
$$

where we have used $g((\nabla Q)(X, Y), Z) = g(Y, (\nabla Q)(X, Z))$, $X, Y, Z \in \mathfrak{X}(M)$, which follows from the symmetry of the operator Q, and Equation (2.3) with $H_s = 0$ for constant scalar curvature. Now, using

$$
\begin{aligned}
\sum_i g((\nabla Q)(e_i, Ae_i), \operatorname{gradf}) &= \sum_i g((\nabla Q)(e_i, \operatorname{gradf}), Ae_i) \\
&= \sum_i g(\nabla_{e_i} Q(\operatorname{gradf}), Ae_i) - \sum_i g(\nabla_{e_i} \operatorname{gradf}, Q(Ae_i)) \\
&= \sum_i g(\nabla_{e_i} Q(\operatorname{gradf}), Ae_i) - \sum_i \operatorname{Ric}(Ae_i, Ae_i)
\end{aligned}
$$

in the Equation (3.8), we arrive at

$$
\begin{aligned}
\operatorname{div}((\nabla Q)(\operatorname{gradf}, \operatorname{gradf})) &= 2\sum_i g(\nabla_{e_i} Q(\operatorname{gradf}), Ae_i) - 2\sum_i \operatorname{Ric}(Ae_i, Ae_i) \\
&\quad - \sum_i g(F(\operatorname{gradf}, e_i), Ae_i) - \sum_i g(F(Ae_i, e_i), \operatorname{gradf}) \\
&\quad + \|Q(\operatorname{gradf})\|^2 - \sum R(e_i, \operatorname{gradf}; \operatorname{gradf}, Q(e_i)) .
\end{aligned} \tag{3.9}
$$

Next taking local orthonormal frame $\{e_1, \ldots, e_n\}$ which diagonalizes A, that is $Ae_i = \lambda_i e_i$, we get

$$\sum_i g(F(Ae_i, e_i), \text{gradf}) = \sum_i \lambda_i g(F(e_i, e_i), \text{gradf}) = 0 \tag{3.10}$$

and

$$\begin{aligned}
\sum_i g(F(\text{gradf}, e_i), Ae_i) &= -\sum_i \lambda_i g(F(e_i, \text{gradf}), e_i) \\
&= -\sum_{ij} \lambda_i g((\nabla_{e_j} R)(e_i, \text{gradf})e_j, e_i) \\
&= -\sum_{ij} \lambda_i (\nabla_{e_j} R)(e_i, \text{gradf}; e_j, e_i) \\
&= \sum_{ij} \lambda_i (\nabla_{e_j} R)(e_j, e_i; \text{gradf}; e_i) \qquad (3.11) \\
&= \sum_{ij} \lambda_i g((\nabla_{e_j} R)(e_j, e_i)\,\text{gradf}, e_i) \\
&= \sum_{ij} g((\nabla_{e_j} R)(e_j, e_i)\,\text{gradf}, Ae_i)\,.
\end{aligned}$$

Thus, using Equations (3.2), (3.10), (3.9), and (3.11) we arrive at

$$\begin{aligned}
&\text{div}((\nabla Q)(\text{gradf}, \text{gradf})) - 2\,\text{div}(A(Q(\text{gradf})) = 2\,\text{Ric}(\text{gradf}, \text{grad}\,\Delta f) - \|Q(\text{gradf})\|^2 \\
&\quad - 2\sum_i \text{Ric}(Ae_i, Ae_i) - \sum_i R(e_i, \text{gradf}; \text{gradf}, Q(e_i)) - \sum_{ij} g((\nabla_{e_j} R)(e_j, e_i)\,\text{grad}\ f; Ae_i)).
\end{aligned}$$

Integrating this equation and using Proposition 3.2, we get

$$\begin{aligned}
\int_M \Bigg\{ & 2\,\text{Ric}(\text{gradf}, \text{grad}\,\Delta f) - \|Q(\text{gradf})\|^2 \\
& - 2\sum_i \text{Ric}(Ae_i, Ae_i) - \sum_i R(e_i, \text{gradf}; \text{gradf}, Q(e_i)) \\
& - \sum_{ij} R(e_j, e_i; Ae_i, Ae_j) + \frac{1}{2}\,\|R_{\text{gradf}}\|^2 \Bigg\}\, dv = 0\,.
\end{aligned} \tag{3.12}$$

Using a similar technique as was used in the proof of the Proposition 3.1, we get

$$\begin{aligned}
\sum_i \text{Ric}(Ae_i, Ae_i) - \sum_{ij} R(e_j, e_i; Ae_i, Ae_j) &= \sum_{ij} R(e_j, Ae_i; Ae_i, e_j) \\
- \sum_{ij} R(e_j, e_i; Ae_i, Ae_j) &= \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij}\,,
\end{aligned}$$

which together with (3.12) proves the proposition. □

Lemma 3.4. *Let (M, g) be a compact Riemannian manifold. Then for $f \in C^\infty(M)$,*

$$\int_M \left\{ (\Delta f)^2 - g(\text{grad}\ f, \text{grad}\,\Delta f) \right\} dv = 0\,.$$

Proof. Integrating the equation $\Delta(f\Delta f) = f\Delta^2 f + (\Delta f)^2 - 2g(\text{grad}f, \text{grad}\,\Delta f)$ and using the fact that Δ is self-adjoint we get the result. □

Lemma 3.5. *Let (M, g) be a compact Riemannian manifold. Then for $f \in C^\infty(M)$,*

$$\int_M \left\{\|A\|^2 - (\Delta f)^2 + \text{Ric}(\text{grad}f, \text{grad}f)\right\} dv = 0$$

where $A(X) = \nabla_X \,\text{grad}f$.

Proof. Using Equation (2.6), we compute

$$\begin{aligned}\text{div}(A(\text{grad}f)) &= \sum_i g(\nabla_{e_i} A(\text{grad}f), e_i) = \sum_i g((\nabla A)(e_i, \text{grad}f) + A^2(e_i), e_i) \\ &= g(\text{grad}f, \sum_i (\nabla A)(e_i, e_i)) + \|A\|^2 .\end{aligned}$$

Integrating above equation and using Lemmas 2.3 and 3.4, we get the desired result. □

As a consequence of Lemma 3.5, for $f \in C^\infty(M)$ with $\Delta f = \lambda_1 f$ where λ_1 is the first nonzero eigenvalue of the Laplacian of the Riemannian manifold (M, g), we have

$$\int_M \left\{\|A\|^2 + \left(\text{Ric}(\text{grad}f, \text{grad}f) - \lambda_1 \|\text{grad}f\|^2\right)\right\} dv = 0$$

where we have also used Lemma 3.4. The above integral implies the following:

Corollary 3.6. *There does not exist a compact Riemannian manifold (M, g) whose Ricci curvature satisfies* Ric $\geq \lambda_1$*, where λ_1 is the first nonzero eigenvalue of the Laplacian on M.*

Proof. For, otherwise this would mean from above integral formula that $\|A\| = 0$, which would imply $A = 0$, that is, $\Delta f = 0$, which is impossible as λ_1 is nonzero eigenvalue. □

Lemma 3.7. *Let (M, g) be an n-dimensional compact and connected Riemannian manifold. Then for a $f \in C^\infty(M)$,*

$$\|\nabla A\|^2 \geq \frac{1}{n} \|\text{grad}\,\Delta f\|^2$$

and for a positively curved M, the equality holds if and only if

$$A(X) = -\frac{1}{n}(\Delta f)X, \ X \in \mathfrak{X}(M) ,$$

where $A(X) = \nabla_X \,\text{grad}f$.

Proof. Define a symmetric tensor field B by

$$B(X, Y) = g(AX, Y) + \frac{1}{n}(\Delta f)g(X, Y), \quad X, Y \in \mathfrak{X}(M) .$$

Then we get

$$(\nabla B)(X, Y, Z) = g((\nabla A)(X, Y), Z) + \frac{1}{n} X(\Delta f)g(Y, Z), \quad X, Y, Z \in \mathfrak{X}(M) .$$

This gives

$$\begin{aligned}\|\nabla B\|^2 &= \sum_{ijk}\left[(\nabla B)(e_i,e_j,e_k)\right]^2 = \|\nabla A\|^2 + \frac{1}{n}\|\mathrm{grad}\,\Delta f\|^2 \\ &\quad + \frac{2}{n}\sum_{ij} g((\nabla A)(e_i,e_j),e_j)g(\mathrm{grad}\,\Delta f, e_i)\,.\end{aligned}$$

Now use Equation (2.5) to get

$$\begin{aligned}\|\nabla B\|^2 &= \|\nabla A\|^2 + \frac{1}{n}\|\mathrm{grad}\,\Delta f\|^2 \\ &\quad + \frac{2}{n}\sum_{ij} g((\nabla A)(e_j,e_i) + R(e_i,e_j)\,\mathrm{grad} f, e_j)g(\mathrm{grad}\,\Delta f, e_i) \\ &= \|\nabla A\|^2 + \frac{1}{n}\|\mathrm{grad}\,\Delta f\|^2 + \frac{2}{n}\sum_{ij}\left[g(e_j,(\nabla A)(e_j,e_j))g(\mathrm{grad}\,\Delta f, e_i)\right. \\ &\quad \left. - \mathrm{Ric}(e_i, \mathrm{grad}\ f)g(\mathrm{grad}\,\Delta f, e_i)\right]\,.\end{aligned}$$

Using Lemma 2.3, we get

$$\|\nabla B\|^2 = \|\nabla A\|^2 + \frac{1}{n}\|\mathrm{grad}\,\Delta f\|^2 - \frac{2}{n}\|\mathrm{grad}\,\Delta f\|^2 = \|\nabla A\|^2 - \frac{1}{n}\|\mathrm{grad}\,\Delta f\|^2$$

which proves the inequality $\|\nabla A\|^2 \geq \frac{1}{n}\|\mathrm{grad}\,\Delta f\|^2$. If the equality holds, we should have $\nabla B = 0$ for the symmetric tensor field B and as M is irreducible (M being of positive curvature), we get $B(X,Y) = ag(X,Y)$, for some constant a. This together with the definition of B gives $A(X) = (a - \frac{1}{n}\Delta f)(X)$. Since $\mathrm{tr}\,.A = -\Delta f$, we get $-\Delta f = (na - \Delta f)$, which implies $a = 0$, that is $A(X) = -\frac{1}{n}(\Delta f)X$, $X \in \mathfrak{X}(M)$. □

Lemma 3.8. *Let* $\{e_1,\dots,e_n\}$ *be a local orthonormal frame on an* n*-dimensional compact Riemannian manifold* (M,g) *which diagonalizes* A *with* $A(e_i) = \lambda_i e_i$, *where* $A(X) = \nabla_X$ gradf, *for a* $f \in C^\infty(M)$. *Then*

$$\int_M \left\{\sum_{i<j}(\lambda_i - \lambda_j)^2\right\} dv = (n-1)\int_M (\Delta f)^2\, dv - n\int_M \mathrm{Ric}(\mathrm{grad} f, \mathrm{grad} f)\, dv\,.$$

Proof. We have

$$\begin{aligned}\sum_{ij}(\lambda_i-\lambda_j)^2 &= \sum_{ij}\lambda_i^2 + \sum_{ij}\lambda_j^2 - 2\sum_{ij}\lambda_i\lambda_j \\ &= n\,\|A\|^2 + n\,\|A\|^2 - 2\sum_j(-\Delta f)\lambda_j = 2n\,\|A\|^2 - 2(\Delta f)^2\end{aligned}$$

and $\sum_{ij}(\lambda_i-\lambda_j)^2 = 2\sum_{i<j}(\lambda_i-\lambda_j)^2$. Thus, $\sum_{i<j}(\lambda_i-\lambda_j)^2 = n\,\|A\|^2 - (\Delta f)^2$.

Integrating this equation and using Lemma 3.5, we get the result. □

From Propositions 3.1 and 3.3, we have the following.

Lemma 3.9. *Let (M, g) be a compact Riemannian manifold of constant scalar curvature. Then for $f \in C^\infty(M)$ and $A(X) = \nabla_X$ gradf,*

$$\int_M \Big\{ \|\nabla A\|^2 - \|\text{grad}\,\Delta f\|^2 + 4\,\text{Ric}(\text{gradf}, \text{grad}\,\Delta f) - 2\,\|Q(\text{gradf})\|^2 - 3\sum_i \text{Ric}(Ae_i, Ae_i)$$
$$- \sum_i R(e_i, \text{gradf}; \text{gradf}, Q(e_i)) + 2\sum_{i<j}(\lambda_i - \lambda_j)^2 K_{ij} \Big\}\, dv = 0$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M, λ_i is an eigenvalue of A and K_{ij} is the sectional curvature of the plane section spanned by $\{e_i, e_j\}$.

As a consequence of Lemma 3.9, we have the following corollary which is the main result in [7].

Corollary 3.10. *Let (M, g) be an n-dimensional compact and connected Einstein manifold, $n \geq 3$, whose sectional curvatures are bounded below by a constant k_0. Then either M is isometric to a sphere or else each nonzero eigenvalue of the Laplacian satisfies $\lambda > 2nk_0$.*

Proof. Since M is an Einstein manifold we have $\text{Ric}(X, Y) = \frac{S}{n} g(X, Y)$, where S is the scalar curvature which is a constant. Then for $\Delta f = \lambda f$, $\lambda > 0$ we have $\text{Ric}(\text{gradf}, \text{grad}\,\Delta f) = \frac{\lambda S}{n}\,\|\text{gradf}\|^2$, $\|Q(\text{gradf})\|^2 = \frac{S^2}{n^2}\,\|\text{gradf}\|^2$,

$$\sum_i \text{Ric}(Ae_i, Ae_i) = \frac{S}{n}\,\|A\|^2, \sum_i R(e_i, \text{gradf}, \text{gradf}, Q(e_i)) = \frac{S^2}{n^2}\,\|\text{gradf}\|^2$$

and consequently Lemma 3.9 gives

$$\int_M \Big\{ \|\nabla A\|^2 - \frac{1}{n}\,\|\text{grad}\,\Delta f\|^2 \Big\}\, dv + \int_M \Big\{ -\frac{(n-1)}{n}\lambda(\Delta f)^2 + \left(\frac{4\lambda S}{n} - \frac{3S^2}{n^2}\right)\|\text{gradf}\|^2$$
$$- \frac{3S}{n}\,\|A\|^2 + 2\sum_{\alpha<i}(\lambda_\alpha - \lambda_i)^2 K_{\alpha i} \Big\}\, dv = 0$$

where we have used Lemma 3.4. Now, using Lemmas 3.5 and 3.8 in the above equation we get

$$\int_M \Big\{ \|\nabla A\|^2 - \frac{1}{n}\,\|\text{grad}\,\Delta f\|^2 \Big\}\, dv + \left(2k_0 - \frac{\lambda}{n}\right)\int_M \Big\{ \sum_{\alpha<i}(\lambda_\alpha - \lambda_i)^2 \Big\}\, dv \leq 0\,.$$

If $\left(2k_0 - \frac{\lambda}{n}\right) \geq 0$, that is $k_0 \geq \frac{\lambda}{2n} > 0$, then above equation together with Lemma 3.7 gives $H_f(X, Y) = \frac{-\lambda f}{n} g(X, Y)$, which by Obata's theorem (cf. [5]) implies that M is isometric to a sphere. Otherwise, $\lambda > 2nk_0$. □

4. Proof of the main theorem

Let (M, g) be the Riemannian manifold as given in the statement of the theorem. Then for $f \in C^\infty(M)$ with $\Delta f = \lambda f$, $\lambda > 0$, Lemma 3.9 gives

$$\int_M \left\{ \|\nabla A\|^2 - \frac{1}{n}\|\text{grad}\,\Delta f\|^2 \right\} dv + \int_M \left\{ -\frac{(n-1)}{n}\lambda(\Delta f)^2 + 4\lambda\,\text{Ric}(\text{gradf}, \text{gradf}) \right.$$
$$- 2\,\|Q(\text{gradf})\|^2 - 3\sum_i \text{Ric}(Ae_i, Ae_i) - \sum R(e_i, \text{gradf}; \text{gradf}, Q(e_i))$$
$$\left. + 2\sum_{i<j}(\lambda_i - \lambda_j)^2 K_{ij} \right\} dv = 0 \tag{4.1}$$

where we have used Lemma 3.4. Since $(n-1)\alpha \leq \text{Ric} \leq (n-1)\alpha\delta$, for a local orthonormal frame $\{e_1, \ldots, e_n\}$ which diagonalizes Q with $Q(e_i) = \mu_i e_i$, we have $\mu_i = \text{Ric}(e_i, e_i)$, that is, $\mu_i \leq (n-1)\alpha\delta$ and consequently,

$$\begin{aligned} \sum_i R(e_i, \text{gradf}; \text{gradf}, Q(e_i)) &= \sum_i \mu_i R(e_i, \text{gradf}; \text{gradf}, e_i) \\ &\leq (n-1)\alpha\delta \sum_i R(e_i, \text{gradf}; \text{gradf}, e_i)\,. \end{aligned}$$

Thus,

$$\sum_i R(e_i, \text{gradf}; \text{gradf}, Q(e_i)) \leq (n-1)\alpha\delta\,\text{Ric}(\text{gradf}, \text{gradf})\,. \tag{4.2}$$

Next we use Lemma 3.5 to arrive at

$$\begin{aligned} \int_M \left\{ \sum_i \text{Ric}(Ae_i, Ae_i) \right\} dv &\leq (n-1)\alpha\delta \int_M \|A\|^2\,dv = (n-1)\alpha\delta \int_M (\Delta f)^2\,dv \\ &\quad - (n-1)\alpha\delta \int_M \text{Ric}(\text{gradf}, \text{gradf})\,dv\,. \end{aligned} \tag{4.3}$$

For estimating the term $\|Q(\text{gradf})\|^2$, we have with $Q(e_i) = \mu_i e_i$ and $\mu_i = \text{Ric}(e_i, e_i)$ that is,

$$\|Q(\text{gradf})\|^2 = \sum_i g(Q(\text{gradf}), e_i)^2 = \sum_i \mu_i^2 g(\text{gradf}, e_i)^2\,.$$

Since $0 < (n-1)\alpha \leq \mu_i \leq (n-1)\alpha\delta$, the above equation gives

$$\begin{aligned} \|Q(\text{gradf})\|^2 &\leq (n-1)^2\alpha^2\delta^2 \sum_i g(\text{gradf}, e_i)^2 \\ &= (n-1)^2\alpha^2\delta^2\,\|\text{gradf}\|^2\,. \end{aligned} \tag{4.4}$$

Also we have using Lemma 3.8 and sectional curvatures are being bounded below by α, that

$$\begin{aligned} \int_M \left\{ \sum_{i<j} (\lambda_i - \lambda_j)^2 \right\} dv &\geq (n-1)\alpha \int_M (\Delta f)^2\,dv \\ &\quad - n\alpha \int_M \text{Ric}(\text{gradf}, \text{gradf})\,dv\,. \end{aligned} \tag{4.5}$$

Using inequalities (4.2)–(4.5) in Equation (4.1) together with bounds on the Ricci curvature, we get

$$\int_M \left\{ \|\nabla A\|^2 - \frac{1}{n} \|\text{grad}\, \Delta f\|^2 \right\} dv + \int_M \left\{ -\frac{(n-1)}{n}\lambda - 3(n-1)\alpha\delta + 2(n-1)\alpha \right\} (\Delta f)^2\, dv$$
$$+ \int_M \left\{ 4\lambda(n-1)\alpha - 2(n-1)^2\alpha^2\delta^2 - 2(n-1)\alpha^2\delta \right\} \|\text{gradf}\|^2\, dv \ \le 0\,.$$

Using Lemma 3.4 in above inequality we arrive at

$$\int_M \left\{ \|\nabla A\|^2 - \frac{1}{n} \|\text{grad}\, \Delta f\|^2 \right\} dv - \frac{(n-1)}{n} E(\lambda, \alpha, \delta) \int_M \|\text{gradf}\|^2\, dv \le 0 \qquad (4.6)$$

where the constant $E(\lambda, \alpha, \delta)$ is $E = \lambda^2 - 3n\alpha(2-\delta)\lambda + 2n\alpha^2\delta(1 + (n-1)\delta)$. If $E \le 0$, then inequality (4.6) together with Lemma 3.7, gives $A = -\frac{\lambda f}{n} I$, (as $\alpha > 0$). Thus, $H_f(X, Y) = -\frac{\lambda f}{n} g(X, Y)$, and Obata's theorem implies that M is isometric to an n-sphere $S^n(\alpha)$, as $\alpha \le c \le \alpha\delta$ and $\delta = 1$. The other option is $E > 0$ and this proves the theorem.

As a consequence of our theorem, we recover the theorem of Simon [7].

Corollary 4.1. *Let (M, g) be an n-dimensional compact and connected Einstein manifold $n \ge 3$, then either M is isometric to the n-sphere $S^n(\alpha)$ or else each nonzero eigenvalue λ of the Laplacian satisfies $\lambda > 2n\alpha$, where $\alpha > 0$ is the infimum of the sectional curvatures of M.*

Proof. Since an Einstein manifold has constant scalar curvature and $\delta = 1$, by above theorem either M is isometric to $S^n(\alpha)$ or else $\lambda^2 - 3n\alpha\lambda + 2n^2\alpha^2 > 0$. This implies $(\lambda - n\alpha)(\lambda - 2n\alpha) > 0$ together with Lichnorowicz theorem $\lambda > n\alpha$ (for the case that M is not isometric to a sphere) that $\lambda > 2n\alpha$. □

5. Characterization of spheres

In this section we use the length $\|R_X\|$ of the curvature tensor field in the direction of $X \in \mathfrak{X}(M)$ and the integral formulas in Section 3 to obtain a characterization of spheres. Note that for $S^n(c)$, we have $\|R_X\|^2 = 2(n-1)c^2 \|X\|^2$ and the first nonzero eigenvalue λ_1 of the Laplacian is given by $\lambda_1 = nc$, thus,

$$\|R_X\|^2 = \frac{(n-1)}{n}(2\lambda_1 c)\, \|X\|^2 = \frac{(n-1)}{n}(\lambda_1 c)\left[(n+2) - \frac{\lambda_1}{c}\right] \|X\|^2$$

holds. This raises a question: Is a compact and connected n-dimensional positively curved Riemannian manifold of constant scalar curvature satisfying

$$\|R_X\|^2 = \frac{(n-1)}{n}(\lambda_1 k_0)\left[(n+2) - \frac{\lambda_1}{k_0}\right] \|X\|^2\,,$$

k_0 being infimum of the sectional curvatures, isometric to $S^n(k_0)$? Indeed, the answer to this question is in affirmative and is a consequence of the following.

Theorem 5.1. *Let (M, g) be an n-dimensional compact and connected Riemannian manifold of constant scalar curvature whose Ricci curvature satisfies* $\text{Ric} \ge (n-1)\alpha$ *for a constant $\alpha > 0$.*

If there exists a nonzero eigenvalue λ *of the Laplacian satisfying*

$$\|R_X\|^2 \leq \frac{n-1}{n}(\lambda\alpha)\left[(n+2) - \frac{\lambda}{\alpha}\right]\|X\|^2, \qquad X \in \mathfrak{X}(M),$$

then M *is isometric to* $S^n(\alpha)$.

Proof. Let $f \in C^\infty(M)$ be such that $\Delta f = \lambda f$, $\lambda > 0$. We subtract integral in the Proposition 3.3 from that in Proposition 3.1 to arrive at

$$\int_M \left\{ \|\nabla A\|^2 - \|\text{grad}\,\Delta f\|^2 + 3\sum_i \text{Ric}(Ae_i, Ae_i) \right.$$
$$\left. + \sum_i R(e_i, \text{gradf}; \text{gradf}, Q(e_i)) - \left\|R_{\text{gradf}}\right\|^2 \right\} dv = 0. \tag{5.1}$$

The assumption on the Ricci curvature together with Lemma 3.5 gives

$$\int_M \left\{\sum_i \text{Ric}(Ae_i, Ae_i)\right\} dv \geq (n-1)\alpha \int_M \left\{(\Delta f)^2 - \text{Ric}(\text{gradf}, \text{gradf})\right\} dv. \tag{5.2}$$

Similarly, we get for $Q(e_i) = \mu_i e_i$, $\mu_i = \text{Ric}(e_i, e_i) \geq (n-1)\alpha$ that

$$\int_M \left\{\sum_i R(e_i, \text{gradf}; \text{gradf}, Q(e_i))\right\} dv \geq (n-1)\alpha \int_M \text{Ric}(\text{gradf}, \text{gradf})\, dv. \tag{5.3}$$

Thus, using (5.2) and (5.3) in (5.1), we arrive at

$$\int_M \left\{\|\nabla A\|^2 - \frac{1}{n}\|\text{grad}\,\Delta f\|^2\right\} dv + \int_M \left\{\frac{(n-1)}{n}(-\lambda + 3\alpha n)(\Delta f)^2 \right.$$
$$\left. -2(n-1)\alpha\, \text{Ric}(\text{gradf}, \text{gradf}) - \left\|R_{\text{gradf}}\right\|^2 \right\} dv \leq 0.$$

Next we use Lemma 3.8 in above inequality to get

$$\int_M \left\{\|\nabla A\|^2 - \frac{1}{n}\|\text{grad}\,\Delta f\|^2\right\} dv + \int_M \left\{\frac{(n-1)}{n}(-\lambda + (n+2)\alpha)(\Delta f)^2 \right.$$
$$\left. + \frac{2(n-1)}{n}\alpha \sum_{i<j}(\lambda_i - \lambda_j)^2 - \left\|R_{\text{gradf}}\right\|^2 \right\} dv \leq 0. \tag{5.4}$$

We use Lemma 3.4, to get

$$\int_M (\Delta f)^2\, dv = \lambda \int_M \|\text{gradf}\|^2\, dv$$

and use this in (5.4) to arrive at

$$\int_M \left\{\|\nabla A\|^2 - \frac{1}{n}\|\text{grad}\,\Delta f\|^2\right\} dv + \int_M \left\{\frac{(n-1)}{n}\lambda\alpha\left((n+2) - \frac{\lambda}{\alpha}\right)\|\text{gradf}\|^2 \right.$$
$$\left. - \left\|R_{\text{gradf}}\right\|^2 \right\} dv + \frac{2(n-1)\alpha}{n} \int_M \left\{\sum_{i<j}(\lambda_i - \lambda_j)^2\right\} dv \leq 0.$$

This inequality and the expression for $\|R_{\text{gradf}}\|^2$ as given in the statement of the theorem together with Lemma 3.7 gives $\lambda_1 = \ldots = \lambda_n = \mu$ say, and

$$\frac{(n-1)}{n}\lambda\alpha\left((n+2)-\frac{\lambda}{\alpha}\right)\|\text{gradf}\|^2 = \|R_{\text{gradf}}\|^2 \,. \tag{5.5}$$

Thus, $A = \mu I$, consequently $A = \frac{-\lambda f}{n} I$, which by Obata's theorem implies that M is isometric to a sphere $S^n(c)$. Since for $S^n(c)$, $\|R_{\text{gradf}}\|^2 = 2(n-1)c^2\|\text{gradf}\|^2$ holds, we have from (5.5)

$$-\lambda^2 + (n+2)\alpha\lambda = 2\lambda_1 c$$

where λ_1 is the first nonzero eigenvalue of the Laplacian on $S^n(c)$, and $\lambda_1 = nc$. This gives

$$(n\alpha - \lambda) = 2\left[\left(\frac{\lambda_1}{\lambda}\right)c - \alpha\right] \le 2(c-\alpha)\,.$$

Since $\lambda \ge n\alpha$, we get $c \le \alpha$. However, for $S^n(c)$, Ric $= (n-1)c$ and thus the statement of the theorem implies that $c \ge \alpha$ which proves that $c = \alpha$, and this finishes the proof. □

Theorem 5.2. *Let (M, g) be an n-dimensional compact and connected Riemannian manifold of constant scalar curvature whose sectional curvatures are bounded below by a constant $\alpha > 0$. If there exists a nonzero eigenvalue λ of the Laplacian satisfying $\frac{2}{3}\alpha\delta(n-1)(\delta-1) < \lambda < 2n\alpha$ for a constant $\delta \ge 1$ and the Ricci curvature satisfies*

$$\frac{3(n-1)\alpha\delta\lambda}{3\lambda - 2\alpha\delta(n-1)(\delta-1)} \le \text{Ric} \le (n-1)\alpha\delta$$

then M is isometric to $S^n(\alpha)$.

Proof. Let (M, g) be the Riemannian manifold as given in the statement of the theorem. Then for $f \in C^\infty(M)$ with $\Delta f = \lambda f$, $\lambda > 0$, Equation (4.1) together with Lemma 3.8, gives

$$\begin{aligned}
&\int_M \left\{\|\nabla A\|^2 - \frac{1}{n}\|\text{grad}\,\Delta f\|^2\right\} dv + \int_M \Bigg\{3\lambda\,\text{Ric}(\text{gradf}, \text{gradf}) - 2\,\|Q(\text{gradf})\|^2 \\
&\quad - 3\sum_i \text{Ric}(Ae_i, Ae_i) - \sum R(e_i, \text{gradf}; \text{gradf}, Q(e_i)) \\
&\quad + \sum_{i<j}(\lambda_i - \lambda_j)^2\left(2\alpha - \frac{\lambda}{n}\right)\Bigg\} dv = 0\,.
\end{aligned} \tag{5.6}$$

Since $(n-1)\alpha \le \text{Ric} \le (n-1)\alpha\delta$, $\delta \ge 1$, for a local orthonormal frame $\{e_1, \ldots, e_n\}$ which diagonalizes Q with $Q(e_i) = \mu_i e_i$, we have $\mu_i =$Ric(e_i, e_i), that is, $\mu_i \le (n-1)\alpha\delta$ and consequently using Equations (4.2), (4.4), and the estimate on $\|Q(\text{gradf})\|^2 \le (n-1)^2\alpha^2\delta^2\|\text{gradf}\|^2 \le (n-1)\alpha\delta^2\,\text{Ric}(\text{gradf}, \text{gradf})$ in above equation we arrive at

$$\begin{aligned}
&\int_M \left\{\|\nabla A\|^2 - \frac{1}{n}\|\text{grad}\,\Delta f\|^2\right\} dv \\
&\quad + \int_M \Bigg\{\left(3\lambda - 2(n-1)\alpha\delta^2 + 2(n-1)\alpha\delta\right)\text{Ric}(\text{gradf}, \text{gradf}) \\
&\quad - 3(n-1)\alpha\delta(\Delta f)^2 + \left(2\alpha - \frac{\lambda}{n}\right)\sum_{i<j}(\lambda_i - \lambda_j)^2\Bigg\} dv\Bigg\} \le 0\,.
\end{aligned}$$

The above integral inequality takes the form

$$\int_M \left\{ \|\nabla A\|^2 - \frac{1}{n} \|\mathrm{grad}\, \Delta f\|^2 \right\} dv$$
$$+ (3\lambda - 2(n-1)\alpha\delta(\delta - 1)) \int_M \left\{ \mathrm{Ric}(\mathrm{gradf}, \mathrm{gradf}) \right.$$
$$\left. - \frac{3(n-1)\alpha\delta\lambda}{3\lambda - 2(n-1)\alpha\delta(\delta-1)} \|\mathrm{gradf}\|^2 \right\} dv$$
$$+ \left(2\alpha - \frac{\lambda}{n}\right) \int_M \sum_{i<j} (\lambda_i - \lambda_j)^2 \left. \right\} dv \leq 0 \tag{5.7}$$

where we have used Lemma 3.4. Using given conditions on λ and the Ricci curvature in (5.7) we get as in the proof of Theorem 5.1, that M is isometric to $S^n(c)$ and that

$$\mathrm{Ric}(\mathrm{gradf}, \mathrm{gradf}) = \frac{3(n-1)\alpha\delta\lambda}{3\lambda - 2(n-1)\alpha\delta(\delta-1)} \|\mathrm{gradf}\|^2 .$$

This equation and $\mathrm{Ric}(\mathrm{gradf}, \mathrm{gradf}) = (n-1)c\,\|\mathrm{gradf}\|^2$ for $S^n(c)$, gives

$$3\lambda c - 2(n-1)c\alpha\delta(\delta - 1) = 3\alpha\delta\lambda . \tag{5.8}$$

Since Ric $\leq (n-1)\alpha\delta$ for $S^n(c)$ we get $(n-1)c \leq (n-1)\alpha\delta$ that is, $c \leq \alpha\delta$. Using $c \leq \alpha\delta$ in (5.8) we get $\delta = 1$ and consequently (5.8) gives $c = \alpha$ and we get that M is isometric to $S^n(\alpha)$. □

In [10, p. 676] Yau has suggested a problem "to find constants α and β so that if the Ricci curvature of a compact manifold satisfies $\alpha \leq$ Ric $\leq \beta$, then the manifold admits an Einstein metric." Theorem 5.2 can be considered as a result in this direction.

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Received May 24, 2005

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Communicated by Peter Li