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Minimal hypersurfaces in a nearly Kaehler 6-sphere

Sharief Deshmukh *

Department of Mathematics, College of Science, King Saud University, P.O. Box#2455, Riyadh-11451, Saudi Arabia

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ABSTRACT

In this paper we show that for a compact minimal hypersurface M of constant scalar curvature in the unit sphere S^6 with the shape operator A satisfying $\|A\|^2 > 5$, there exists an eigenvalue $\lambda > 10$ of the Laplace operator of the hypersurface M such that $\|A\|^2 = \lambda - 5$. This gives the next discrete value of $\|A\|^2$ greater than 0 and 5.

1. Introduction

Let M be a compact immersed hypersurface of constant scalar curvature in the unit sphere S^{n+1} and A be its shape operator. In the geometry of minimal hypersurfaces of the unit sphere Chern's conjecture "For compact minimal hypersurfaces of constant scalar curvature in the unit sphere S^{n+1} the set of values of the square of the length of the shape operator $\|A\|^2$ is a discrete set", is well known (cf. [1], p.693). It is known that the first two values of $\|A\|^2$ are 0 and n (cf. [2–5]). In respect of the third value of $\|A\|^2$, Peng and Terng [6] have proved that if $\|A\|^2 > n$, then $\|A\|^2 > n + c(n)$ where $c(n) > \frac{1}{12n}$ is a positive constant. Also for $n = 3$ these authors proved that $\|A\|^2 \geq 6$ and consequently they conjectured that the third value of $\|A\|^2$ should be $2n$. Indeed the immersion $f : SO(3) \to S^4$ of the Lie group $SO(3)$ defined by $f(g) = gBg^{-1}$, where B is a 3×3 diagonal matrix with diagonal $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0$ is a minimal immersion with $\|A\|^2 = 6$ (cf. [7]). Then Yang and Cheng (cf. [8,9]) improved the result of Peng and Terng by proving $c(n) > \frac{2}{7}n - \frac{9}{14}$. These authors in [10] further improved this result by proving that if $\|A\|^2 > n$, then $\|A\|^2 \geq \frac{1}{3}(4n + 1)$. In this paper using the nearly Kaehler structure of S^6, we prove the following.

Theorem. *Let M be a compact minimal hypersurface of constant scalar curvature in the unit sphere S^6. If the shape operator A of M satisfies $\|A\|^2 > 5$, then there exists an eigenvalue $\lambda > 10$ of the Laplace operator on M satisfying $\|A\|^2 = \lambda - 5$.*

2. Preliminaries

It is known that the six-dimensional unit sphere S^6 has a nearly Kaehler structure $(J, \bar{g})$, where J is an almost complex structure defined on S^6 using the vector cross product of purely imaginary Cayley numbers R^7 and $\bar{g}$ is the induced metric

* Tel.: +966 12089016; fax: +966 14676512.
E-mail addresses: shariefd@ksu.edu.sa, shariefd@hotmail.com.

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on S^6 as a hypersurface of R^7. Also S^6 can be expressed as $S^6 = G_2/SU(3)$ a homogeneous almost Hermitian manifold, where G_2 is the compact Lie group of all automorphisms of the Cayley division algebra R^8. Let S^6 be the nearly Kaehler 6-sphere with a nearly Kaehler structure $(J, \overline{g})$, where J is the almost complex structure and $\overline{g}$ is the induced metric on S^6. Then we have

$$\left(\overline{\nabla}_X J\right)(X) = 0, \qquad \overline{g}\,(JX, JY) = \overline{g}\,(X, Y)\,, \quad X, Y \in \mathfrak{X}(S^6), \tag{2.1}$$

where $\overline{\nabla}$ is the Riemannian connection with respect to the almost Hermitian metric $\overline{g}$ and $\mathfrak{X}(S^6)$ is the Lie algebra of smooth vector fields on S^6 (cf. [11,12]). The tensor field G of type (2, 1) defined by $G(X, Y) = (\overline{\nabla}_X J)(Y)$, $X, Y \in \mathfrak{X}(S^6)$ has the properties as described in the following

Lemma 2.1 (*[12]*). (a) $G(X, JY) = -JG(X, Y)$, (b) $G(X, Y) = -G(Y, X)$, (c) $(\overline{\nabla}_X G)(Y, Z) = \overline{g}(Y, JZ)X + \overline{g}(X, Z)JY - \overline{g}(X, Y)JZ$, $X, Y, Z \in \mathfrak{X}(S^6)$.

Let M be an immersed orientable minimal real hypersurface of S^6, ∇ be the Riemannian connection with respect to the induced metric g on M and N be the unit normal vector field. Then we have (cf. [13])

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \overline{\nabla}_X N = -AX, \quad X, Y \in \mathfrak{X}(M), \tag{2.2}$$

where A is the shape operator of the hypersurface M. The Gauss and Codazzi equations for the hypersurface are

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(AY, Z)AX - g(AX, Z)AY, \tag{2.3}$$

$$(\nabla A)\,(X, Y) = (\nabla A)\,(Y, X) \tag{2.4}$$

for $X, Y, Z \in \mathfrak{X}(M)$, where $(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y)$. The Ricci tensor Ric and the scalar curvature S of the hypersurface are given by

$$\mathrm{Ric}(X, Y) = 4g(X, Y) - g(AX, AY), \tag{2.5}$$

$$S = 20 - \|A\|^2, \tag{2.6}$$

where $\|A\|^2 = \mathrm{tr}\, A^2$ is the square of the length of the shape operator of the hypersurface.

Using the almost complex structure J of S^6, we define a unit vector field $\xi \in \mathfrak{X}(M)$ by $\xi = -JN$, with dual 1-form $\eta(X) = g(X, \xi)$. For a $X \in \mathfrak{X}(M)$, we set $JX = \varphi(X) + \eta(X)N$, where $\varphi(X)$ is the tangential component of JX. Then it follows that φ is a (1, 1) tensor field on M. Using $J^2 = -I$, it is easy to see that (φ, ξ, η, g) defines an almost contact metric structure on M, that is

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta\,(\xi) = 1, \qquad \eta \circ \varphi = 0, \qquad \varphi\,(\xi) = 0 \tag{2.7}$$

and $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$, $X, Y \in \mathfrak{X}(M)$. We observe that $G(X, N)$ is orthogonal to N for $X \in \mathfrak{X}(M)$, and consequently $G(X, N) \in \mathfrak{X}(M)$, which enables us to define another (1, 1) tensor field $\overline{\varphi}$ on the hypersurface M, by $\overline{\varphi}(X) = G(X, N)$. Using Lemma 2.1, we immediately get that $(\overline{\varphi}, \xi, \eta, g)$ defines another almost contact metric structure on M, that is

$$\overline{\varphi}^2 = -I + \eta \otimes \xi, \qquad \eta\,(\xi) = 1, \qquad \eta \circ \overline{\varphi} = 0, \qquad \overline{\varphi}\,(\xi) = 0 \tag{2.8}$$

and $g(\overline{\varphi}X, \overline{\varphi}Y) = g(X, Y) - \eta(X)\eta(Y)$, $X, Y \in \mathfrak{X}(M)$. Using Lemma 2.1 and Eq. (2.2), we easily get the following

Lemma 2.2. *Let M be an orientable real hypersurface of S^6. Then the structures (φ, ξ, η, g), $(\overline{\varphi}, \xi, \eta, g)$ on M satisfy*
(i) $(\nabla_X \varphi)(Y) = \eta(Y)AX - g(AX, Y)\xi + G(X, Y)^T$, (ii) $\nabla_X \xi = \varphi AX - \overline{\varphi}X$, (iii) $g(\varphi X, \overline{\varphi}X) = 0$, (iv) $(\nabla_X \overline{\varphi})(Y) = g(X, Y)\xi - \eta(Y)X + G(AX, Y)^T$, $X, Y \in \mathfrak{X}(M)$, *where $G(X, Y)^T$ is the tangential component of $G(X, Y)$ to M.*

Note that φ and $\overline{\varphi}$ are skew symmetric and for a unit vector field e_1 orthogonal to ξ, $\{e_1, \varphi e_1, \xi\}$ is an orthonormal set of vector fields and that if e_2 is a unit vector field orthogonal to $e_1, \varphi e_1$ and ξ, then $\{e_1, \varphi e_1, e_2, \varphi e_2, \xi\}$ (respectively $\{e_1, \overline{\varphi} e_1, e_2, \overline{\varphi} e_2, \xi\}$) forms a local orthonormal frame on the hypersurface M, called an adapted frame. Using an adapted frame together with Lemma 2.2 one immediately concludes the following

Lemma 2.3. *Let M be an orientable real hypersurface of S^6. Then* $\mathrm{div}\,\xi = 0$.

3. Proof of the theorem

Let M be an orientable compact hypersurface of S^6 satisfying the hypothesis of the Theorem. We denote by $\overline{N}$ the unit normal vector field of the unit sphere S^6 in R^7. Choose a constant unit vector field Z on R^7 and define smooth functions $f, h : M \to R$ by $f = \langle Z, N\rangle$, $h = \langle Z, \overline{N}\rangle$, where $\langle, \rangle$ is the Euclidean metric on R^7.Then we can express the restriction of Z to M as

$$Z = u + fN + h\overline{N}, \tag{3.1}$$

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where $u \in \mathfrak{X}(M)$. Using Eq. (2.2) and similar equation for the hypersurface S^6 of R^7 in above equation, we immediately get

$$\nabla_X u = fAX - hX, \qquad \nabla f = -Au, \qquad \nabla h = u, \tag{3.2}$$

where $X \in \mathfrak{X}(M)$ and ∇f, ∇h are gradients of the functions f, h respectively. Now for the function $\bar{h} = g(\xi, u)$, we use Lemma 2.2 and Eq. (3.2) to compute the gradient $\nabla\bar{h}$ as

$$\nabla\bar{h} = -A\varphi u + \bar{\varphi}u + fA\xi - h\xi. \tag{3.3}$$

Choosing a local orthonormal frame $\{e_1, \ldots, e_5\}$ on M and using the fact that M is minimal we use Lemma 2.2 and Eq. (3.2) to compute

$$\begin{aligned}\operatorname{div}(A\varphi u) &= \sum_i g\left(\left(\nabla_{e_i}\varphi\right)(u) + \varphi(fAe_i - he_i), Ae_i\right)\\ &= \bar{h}\|A\|^2 - g(Au, A\xi),\end{aligned} \tag{3.4}$$

where we used skew symmetry of φ, $\sum_i g(\varphi e_i, Ae_i) = 0$ and $\sum_i g(G(e_i, u), Ae_i) = 0$ (which follow by taking a local orthonormal frame that diagonalizes A). Similarly we compute

$$\operatorname{div}(\bar{\varphi}u) = -\sum_i g(fAe_i - he_i, \bar{\varphi}e_i) - \sum_i g\left(u, \left(\nabla_{e_i}\bar{\varphi}\right)(e_i)\right) = -4\bar{h}, \tag{3.5}$$

$$\operatorname{div}(fA\xi) = -g(Au, A\xi) + \sum_i fg((\varphi Ae_i - \bar{\varphi}e_i), Ae_i) = -g(Au, A\xi), \tag{3.6}$$

$$\operatorname{div}(h\xi) = \bar{h}, \tag{3.7}$$

where we used Lemma 2.3 in the last equation. Using Eqs. (3.4), (3.5), (3.6) and (3.7) in Eq. (3.3) to compute the Laplacian $\Delta\bar{h}$, arrive at

$$\Delta\bar{h} = -(5 + \|A\|^2)\bar{h}. \tag{3.8}$$

We claim that there exists a constant unit vector field Z on R^7 with respect to which $\bar{h}$ is not a constant. If $\bar{h}$ is a constant, then in view of Eq. (3.8), we get $\bar{h} = 0$. This would imply ξ is orthogonal to u and consequently $\langle Z, \xi\rangle = 0$. If this holds for each constant unit vector field on R^7 that will imply $\xi = 0$ a contradiction as ξ is a unit vector field. Hence there exists a constant vector field Z with respect to which $\bar{h}$ is not a constant. Eq. (3.8) implies that $\bar{h}$ is an eigenfunction of the Laplace operator Δ corresponding to the eigenvalue $\lambda = 5 + \|A\|^2 > 10$ and this proves that $\|A\|^2 = \lambda - 5$.

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