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Submanifolds of positive Ricci curvature in a Euclidean space

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Abstract In this paper we study the role of constant vector fields on a Euclidean space R^{n+p} in shaping the geometry of its compact submanifolds. For an n-dimensional compact submanifold M of the Euclidean space R^{n+p} with mean curvature vector field H and a constant vector field ξ on R^{n+p}, the smooth function $\varphi = \langle H, \xi\rangle$ is used to obtain a characterization of sphere among compact submanifolds of positive Ricci curvature (cf. main Theorem).

Keywords Submanifolds in a Euclidean space · Ricci curvature · Mean curvature vector field

Mathematics Subject Classification (2000) 53C20 · 53C40

1 Introduction

Given an n-dimensional submanifold M of a Euclidean space R^{n+p} with immersion $\psi : M \to R^{n+p}$, the position vector field ψ has been quite extensively used in studying the geometry of the submanifold ([2,7,12,13]). In particular for compact hypersurfaces it is used to obtain a very useful tool the Minkoswki's formula and it has been used for obtaining very fruitful information about the geometry of the hypersurfaces (cf. [3–6,8,9,15,16]). However constant vector fields on the ambient Euclidean space R^{n+p} which are in abundance are not so far much used in studying the geometry of the submanifolds in the Euclidean space R^{n+p}. Let $\psi : M \to R^{n+p}$ be an n-dimensional compact submanifold with mean curvature vector field H. Then for a constant vector field $\xi \in \mathfrak{X}(R^{n+p})$, we write $\xi = u + \varsigma$, u and ς being tangential and normal components of ξ restricted to M and define a smooth function $\varphi = \langle H, \xi\rangle$, where $\langle , \rangle$ is the

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Euclidean inner product and $\mathfrak{X}(R^{n+p})$ is the Lie algebra of smooth vector fields on R^{n+p}. There are two operators A, B on the submanifold M naturally associated to this constant vector field ξ defined by $A = A_\varsigma$ the Weingarten map with respect to the normal vector field ς and B is the symmetric operator associated to the Hessian H_φ of the function φ by $H_\varphi(X, Y) = g(BX, Y), X, Y \in \mathfrak{X}(M)$, where g is the induced metric on the submanifold. For this pair of constant vector field ξ and the submanifold M we define a number $\mu(\xi, M)$ as

$$\mu(\xi, M) = \int_M \left\{(n-1)\,\|B + cA\|^2 - 2nc\left(\operatorname{tr} AB + \|\nabla\varphi\|^2\right)\right\} \mathrm{d}v,$$

where $c = \inf \frac{1}{n-1}\operatorname{Ric}$, Ric being the Ricci curvature of the submanifold M and $\nabla\varphi$ is the gradient of the smooth function φ. This number plays an important role in our study. The motivation of the choice of this number comes from the submanifold $S^n(c)$ of constant curvature c. The second fundamental form of the submanifold $\psi : S^n(c) \to R^{n+p}$ satisfies $h(X, Y) = g(X, Y)H$, $\|H\|^2 = c$, and consequently we get $A = \varphi I$, $B = -c\varphi I$ and $\nabla\varphi = -cu$ and that φ satisfies $\Delta\varphi = -nc\varphi$, that is φ is an eigenfunction of the Laplacian operator Δ with first nonzero eigenvalue nc. Thus by minimum principle one has $\int_{S^n(c)} \|\nabla\varphi\|^2 \mathrm{d}v = nc\int_{S^n(c)} \varphi^2 \mathrm{d}v$. Therefore we conclude that for the Sphere $S^n(c)$, the number $\mu(\xi, M) = 0$. This example allows us to ask the question: "Does an n-dimensional compact submanifold of positive curvature in the Euclidean space R^{n+p} satisfying $\mu(\xi, M) = 0$ for a constant vector field $\xi \in \mathfrak{X}(R^{n+p})$ with non constant smooth function φ necessarily a $S^n(c)$?". We answer this question in affirmative and indeed prove the following:

Theorem 1.1 *Let M be an n-dimensional compact and connected submanifold of the Euclidean space R^{n+p} with mean curvature vector H. Let $\inf\frac{1}{n-1}\operatorname{Ric} = c > 0$ and $\xi \in \mathfrak{X}(R^{n+1})$ be a constant vector field with $\varphi = \langle H, \xi\rangle$. If φ is not a constant and*

$$\mu(\xi, M) = \int_M \left\{(n-1)\,\|B + cA\|^2 - 2nc\left(\operatorname{tr}AB + \|\nabla\varphi\|^2\right)\right\} \mathrm{d}v = 0$$

then $M = S^n(c)$.

In particular case of a compact hypersurface in the Euclidean space the function φ is automatically not a constant as will be seen in the Corollary 3.1, and thus we have the following:

Corollary 1.2 *Let M be an n-dimensional compact and connected hypersurface of the Euclidean space R^{n+1} with mean curvature vector H. Let $\inf\frac{1}{n-1}\operatorname{Ric} = c > 0$ and $\xi \in \mathfrak{X}(R^{n+1})$ be a constant vector field with $\varphi = \langle H, \xi\rangle$. If*

$$\mu(\xi, M) = \int_M \left\{(n-1)\,\|B + cA\|^2 - 2nc\left(\operatorname{tr}AB + \|\nabla\varphi\|^2\right)\right\} \mathrm{d}v = 0$$

then $M = S^n(c)$.

It is worth noting that for a compact submanifold M in the Euclidean space R^{n+p} the mean curvature vector field $H \neq 0$ and consequently there exist constant vector fields ξ (for instance, coordinate vector fields) on R^{n+p} with respect to which φ is not constant (as φ constant implies $\varphi = 0$ by Lemma 2.1 in Sect. 2 and if all φ 's corresponding to coordinate vector fields are constant that would render $H = 0$, which is not true for compact submanifolds in R^{n+p}), thus there are non constant functions φ for compact submanifolds in R^{n+p} as required in above Theorem.

2 Preliminaries

Let $\psi : M \to R^{n+p}$ be an n-dimensional immersed submanifold in the Euclidean space R^{n+p}. We denote by $\langle,\rangle$ and $\overline{\nabla}$ the Euclidean metric and the Euclidean connection on R^{n+p}. Let g and ∇ be the induced metric and the Riemannian connection on the submanifold M. Then we have the following fundamental equations for the submanifold

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \overline{\nabla}_X N = -A_N(X) + \nabla^{\perp}_X N, \tag{2.1}$$

$X, Y \in \mathfrak{X}(M)$, $N \in \Gamma(\upsilon)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M, $\Gamma(\upsilon)$ is the space of smooth sections of the normal bundle υ of M, h is the second fundamental form, A_N is the Weingarten map with respect to the normal vector field N and $\nabla^{\perp}$ is the connection in the normal bundle υ. We also have the following equations of Gauss and Codazzi for the submanifold

$$R(X,Y;Z,W) = g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)), \tag{2.2}$$

$$(Dh)(X,Y,Z) = (Dh)(Y,Z,X) = (Dh)(Z,X,Y), \tag{2.3}$$

where R is the curvature tensor field of the submanifold M and $(Dh)(X,Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$ for $X, Y, Z \in \mathfrak{X}(M)$. The Ricci tensor Ric of the submanifold is given by

$$\mathrm{Ric}(X,Y) = ng(H, h(X,Y)) - \sum_{i=1}^{n} g(h(X,e_i), h(Y,e_i)), \tag{2.4}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame and $H = \frac{1}{n}\sum h(e_i,e_i)$ is the mean curvature vector field. The Ricci operator Q is a symmetric $(1,1)$ tensor field defined by $\mathrm{Ric}(X,Y) = g(Q(X),Y)$, $X, Y \in \mathfrak{X}(M)$.

For a constant vector field $\xi \in \mathfrak{X}(R^{n+p})$ we write $\xi = u + \varsigma$, where $u \in \mathfrak{X}(M)$ is the tangential component and $\varsigma \in \Gamma(\upsilon)$ is the normal component of ξ. If we denote by $A = A_\varsigma$ the Weingarten map with respect to the normal vector field ς, then using Eq. (2.1) and that the vector field ξ is parallel, we immediately have

$$\nabla_X u = A(X), \quad \nabla^{\perp}_X \varsigma = -h(X,u), \quad X \in \mathfrak{X}(M) \tag{2.5}$$

Define a smooth function $\varphi : M \to R$ by $\varphi = \langle H, \xi \rangle$, then using the fact that ξ is parallel, we immediately have the following:

Lemma 2.1 *For a compact submanifold M of R^{n+p} and the constant vector field $\xi \in \mathfrak{X}(R^{n+p})$*

$$\int_M \varphi \, \mathrm{d}v = 0.$$

Using Codazzi equations and the fact that ξ is parallel, we also get the following

Lemma 2.2 *For an n-dimensional compact submanifold M of R^{n+p} and the constant vector field $\xi \in \mathfrak{X}(R^{n+p})$ we have*

(i) $\operatorname{tr}A = n\varphi$,
(ii) $(\nabla A)(X,Y) - (\nabla A)(Y,X) = R(X,Y)u$,
(iii) $\sum(\nabla A)(e_i, e_i) = n\nabla\varphi + Q(u)$,

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of M and $(\nabla A)(X,Y) = \nabla_X AY - A(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$.

Lemma 2.3 *Let M be an n-dimensional compact submanifold M of R^{n+p} and $\xi \in \mathfrak{X}(R^{n+p})$ be the constant vector field. Then*

$$\int_M \left\{\operatorname{Ric}(u,u) + \|A\|^2 - n^2\varphi^2\right\} \mathrm{d}v = 0$$

Proof Let $\{e_1, \ldots, e_n\}$ be a pointwise constant local orthonormal frame. We compute the divergence of the vector field $A(u)$ as

$$\begin{aligned}\operatorname{div}Au &= \sum_i e_i g(u, Ae_i) = \sum_i g(Ae_i, Ae_i) + \sum_i g(u, (\nabla A)(e_i, e_i)) \\ &= \|A\|^2 + ng(u, \nabla\varphi) + \operatorname{Ric}(u,u), \end{aligned} \tag{2.6}$$

where we have used (iii) of Lemma 2.2. Also using Eq. (2.5) we arrive at

$$g(u, \nabla\varphi) = \operatorname{div}(\varphi u) - n\varphi^2. \tag{2.7}$$

Using Eq. (2.7) in Eq. (2.6) and integrating the resulting equation we get the Lemma.

Lemma 2.4 *Let M be an n-dimensional compact submanifold M of R^{n+p} and $\xi \in \mathfrak{X}(R^{n+p})$ be the constant vector field. Then*

$$\int_M \left\{\operatorname{Ric}(\nabla\varphi, u) + \sum_i g(\nabla_{e_i}\nabla\varphi, Ae_i) + n\|\nabla\varphi\|^2\right\} \mathrm{d}v = 0,$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of M.

Proof Using Eqs. (2.4) and (2.5) we have

$$\begin{aligned}
\mathrm{Ric}(\nabla\varphi, u) &= ng(H, h(\nabla\varphi, u)) - \sum_{i=1}^{n} g(h(\nabla\varphi, e_i), h(u, e_i)) \\
&= -ng(H, \nabla^{\perp}_{\nabla\varphi}\varsigma) + \sum_{i=1}^{n} g(h(\nabla\varphi, e_i), \nabla^{\perp}_{e_i}\varsigma) \\
&= -n\left[\nabla\varphi(\varphi) - g(\nabla^{\perp}_{\nabla\varphi} H, \varsigma)\right] \\
&\quad + \sum_i \left[e_i g(h(\nabla\varphi, e_i), \varsigma) - g(\nabla^{\perp}_{e_i} h(\nabla\varphi, e_i), \varsigma)\right] \\
&= -n\|\nabla\varphi\|^2 + ng(\nabla^{\perp}_{\nabla\varphi} H, \varsigma) + \sum_i e_i g(A(\nabla\varphi), e_i) \\
&\quad - \sum_i g((Dh)(e_i, \nabla\varphi, e_i), \varsigma) - \sum_i g(h(\nabla_{e_i}\nabla\varphi, e_i), \varsigma) \\
&= -n\|\nabla\varphi\|^2 + \mathrm{div} A(\nabla\varphi) - \sum_i g(\nabla_{e_i}\nabla\varphi, Ae_i),
\end{aligned}$$

where we used the fact $\sum_i (Dh)(\nabla\varphi, e_i, e_i) = n\nabla^{\perp}_{\nabla\varphi} H$. The last equation proves the Lemma.

For an n-dimensional compact Riemannian manifold (M, g) and a smooth function $\varphi : M \to R$, the Hessian operator $B : \mathfrak{X}(M) \to \mathfrak{X}(M)$ corresponding to the Hessian H_φ of the function φ is defined by $H_\varphi(X, Y) = g(BX, Y)$, $X, Y \in \mathfrak{X}(M)$. Then the operator B has the following properties:

Lemma 2.5 (i) $\mathrm{tr}B = \Delta\varphi$,
(ii) $(\nabla B)(X, Y) - (\nabla B)(Y, X) = R(X, Y)\nabla\varphi$,
(iii) $\sum(\nabla B)(e_i, e_i) = \nabla(\Delta\varphi) + Q(\nabla\varphi)$,
where $\Delta\varphi = \mathrm{div}(\nabla\varphi)$ *is the Laplacian of* φ *and* $\{e_1, \ldots, e_n\}$ *is a local orthonormal frame of* M.

Next, we state the following modified form of the well known Bochner formula

Lemma 2.6 *A smooth function* $\varphi : M \to R$ *on an n-dimensional compact Riemannian manifold* (M, g) *satisfies*

$$\int_M \left\{\mathrm{Ric}(\nabla\varphi, \nabla\varphi) + \|B\|^2 - (\Delta\varphi)^2\right\} \mathrm{d}v = 0.$$

3 Proof of the theorem

Let M be an n-dimensional compact submanifold of the Euclidean space R^{n+p} and $\xi \in \mathfrak{X}(R^{n+p})$ be the constant vector field. We have

$$\mathrm{Ric}(\nabla\varphi + cu, \nabla\varphi + cu) = \mathrm{Ric}(\nabla\varphi, \nabla\varphi) + c^2\mathrm{Ric}(u, u) + 2c\mathrm{Ric}(\nabla\varphi, u)$$

which on integration together with Lemmas 2.3, 2.4 and 2.6 gives

$$\int_M \operatorname{Ric}(\nabla\varphi + cu, \nabla\varphi + cu)\mathrm{d}v = \int_M \Bigg\{ -\|B\|^2 + (\Delta\varphi)^2 - c^2\|A\|^2 + n^2c^2\varphi^2 - 2c\sum_i g(\nabla_{e_i}(\nabla\varphi), Ae_i) - 2nc\|\nabla\varphi\|^2 \Bigg\}\mathrm{d}v$$

Since, $\|B\|^2 \geq \frac{1}{n}(\Delta\varphi)^2$ and $\|A\|^2 \geq n\varphi^2$, we arrive at

$$\int_M \operatorname{Ric}(\nabla\varphi + cu, \nabla\varphi + cu)\mathrm{d}v \leq \int_M \Bigg\{ (n-1)(\|B\|^2 + c^2\|A\|^2) - 2nc\|\nabla\varphi\|^2 - 2c\sum_i g(\nabla_{e_i}(\nabla\varphi), Ae_i) \Bigg\}\mathrm{d}v$$

Next using the fact that $\sum_i g(\nabla_{e_i}(\nabla\varphi), Ae_i) = \operatorname{tr}AB$ and that A, B and $B + cA$ are symmetric operators with $\operatorname{tr}AB = \operatorname{tr}BA$ and consequently that

$$\|B + cA\|^2 = \|B\|^2 + c^2\|A\|^2 + 2c\operatorname{tr}AB$$

in the above inequality we arrive at

$$\int_M \operatorname{Ric}(\nabla\varphi + cu, \nabla\varphi + cu)\mathrm{d}v \leq \mu(\xi, M).$$

Since M has positive Ricci curvature, the condition in the statement of theorem together with above inequality implies that

$$\nabla\varphi = -cu. \tag{3.1}$$

Using Eq. (3.1) in Lemma 2.4 with $\mu(\xi, M) = 0$, we get

$$2n\int_M \Big\{ \operatorname{Ric}(\nabla\varphi, \nabla\varphi) - c(n-1)\|\nabla\varphi\|^2 \Big\}\mathrm{d}v = 0,$$

where we have used $B+cA = 0$ as an outcome of (3.1) and $\sum_i g(\nabla_{e_i}(\nabla\varphi), Ae_i) = \operatorname{tr}AB$. However, as $c = \inf \frac{1}{n-1}\operatorname{Ric}$, $\operatorname{Ric}(\nabla\varphi, \nabla\varphi) \geq (n-1)c\|\nabla\varphi\|^2$, above equation gives

$$\operatorname{Ric}(\nabla\varphi, \nabla\varphi) = (n-1)c\|\nabla\varphi\|^2.$$

This equation together with (3.1) yields $\operatorname{Ric}(u, u) = (n-1)c^{-1}\|\nabla\varphi\|^2$ and using this in Lemma 2.3, we arrive at

$$\int_M \Big\{ (n-1)\|\nabla\varphi\|^2 - n(n-1)c\varphi^2 + c(\|A\|^2 - n\varphi^2) \Big\}\mathrm{d}v = 0. \tag{3.2}$$

Note that Lemma 2.1 together with maximum principle implies that

$$\int_M \|\nabla\varphi\|^2\mathrm{d}v \geq \lambda_1\int_M \varphi^2\mathrm{d}v,$$

where λ_1 is the first nonzero eigenvalue of the Laplacian operator. Thus the Eq. (3.2) takes the form

$$(n-1)(\lambda_1 - nc)\int_M \varphi^2 \mathrm{d}v + c\int_M (\|A\|^2 - n\varphi^2)\mathrm{d}v \leq 0. \tag{3.3}$$

Also by Lichnerowicz Theorem, we know that $\mathrm{Ric} \geq (n-1)c$ implies that $\lambda_1 \geq nc$ and the equality holding if and only if $M = S^n(c)$. Thus the inequality (3.3) implies that $M = S^n(c)$.

Corollary 3.1 *If φ is a constant in the Theorem, then M is a submanifold of the hyperplane R^{n+p-1}.*

Proof Let $\psi : M \to R^{n+p}$ be the immersion of the submanifold M. If φ is a constant, then by Lemma 2.1 $\varphi = 0$ and consequently the constraint $\mu(\xi, M) = 0$ gives $A = 0$. Using $A = 0$ and $\varphi = 0$ in Lemma 2.3, we get $u = 0$. Thus $X\langle\psi,\xi\rangle = \langle X, \varsigma\rangle = 0$, $X \in \mathfrak{X}(M)$. Thus $\langle\psi,\xi\rangle = b$ for a constant b.Then taking the constant vector field $\xi = (a_1, \ldots, a_{n+p})$ we see that $\sum_{i=1}^{n+p} a_i x_i = b$. That is $\psi(M)$ lies in the hyperplane determined by $\sum_{i=1}^{n+p} a_i x_i = b$, where $x_1, \ldots, x_{n+p}$ are the Euclidean coordinates on R^{n+p}. This proves the Corollary.

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