Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
26 (2010), ??–??
www.emis.de/journals
ISSN 1786-0091

CLIFFORD HYPERSURFACES IN A UNIT SPHERE

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ABSTRACT. Let M be a compact Minimal hypersurface of the unit sphere S^{n+1}. In this paper we use a constant vector field on R^{n+2} to characterize the Clifford hypersurfaces $S^l\left(\sqrt{\frac{l}{n}}\right) \times S^m\left(\sqrt{\frac{m}{n}}\right)$, $l+m=n$, in S^{n+1}. We also study compact minimal Einstein hypersurfaces of dimension greater than two in the unit sphere and obtain a lower bound for first nonzero eigenvalue λ_1 of its Laplacian operator.

1. Introduction

Let M be a compact Minimal hypersurface of the unit sphere S^{n+1} and A be its shape operator. In [4], it is shown that if $\|A\|^2 = n$, then the hypersurface is either Veronese surface ($n = 2$) or the Clifford hypersurface $S^l\left(\sqrt{\frac{l}{n}}\right) \times S^m\left(\sqrt{\frac{m}{n}}\right)$, $l+m=n$. For a pair of integers l, m, $l+m=n$, Clifford hypersurface is defined by

$$S^l\left(\sqrt{\frac{l}{n}}\right) \times S^m\left(\sqrt{\frac{m}{n}}\right) = \left\{(x,y) \in R^{l+1} \times R^{m+1} : \|x\|^2 = \frac{l}{n},\ \|y\|^2 = \frac{m}{n}\right\}$$

which is an embedded minimal hypersurface of the unit sphere S^{n+1} of constant scalar curvature and length of its shape operator satisfies $\|A\|^2 = n$. One of the interesting questions is to obtain different characterizations of the Clifford hypersurfaces in the unit sphere S^{n+1}. In this paper we obtain one such characterization for Clifford hypersurfaces among compact minimal hypersurfaces without assuming that they have constant scalar curvature. We denote by N and $\overline{N}$ the unit normal vector field of the minimal hypersurface M in S^{n+1} and that of the unit sphere S^{n+1} in the Euclidean space R^{n+2} respectively. We denote by $\langle , \rangle$ the Euclidean metric on R^{n+2}. One of the main results is the following:

2000 *Mathematics Subject Classification.* 53C40, 53A10.
Key words and phrases. ????

Theorem 1. *Let M be a compact and connected minimal hypersurface of the unit sphere S^{n+1}, $n>2$. Then M is a Clifford hypersurface if and only if there exists a nonzero constant vector field $\mathfrak{a}$ on R^{n+2} such that $\langle \mathfrak{a}, N\rangle = c\left\langle \mathfrak{a}, \overline{N}\right\rangle$ holds for a nonzero constant c.*

In the geometry of minimal hypersurfaces of the unit sphere the Chern's conjecture "For compact minimal hypersurfaces of constant scalar curvature in the unit sphere S^{n+1} the set of values of the square of the length of the shape operator $\|A\|^2$ is a discrete set", is well known (cf. [15, p.693]). It is known that first two values of $\|A\|^2$ are 0 and n (cg. [3, 7, 11]). In respect of the third value of $\|A\|^2$, Peng and Terng [9] have proved that if $\|A\|^2 > n$, then $\|A\|^2 > n + c(n)$ where $c(n) > \frac{1}{12n}$ is a positive constant. Also for $n = 3$ these authors proved that $\|A\|^2 \geq 6$ and consequently they conjectured that the third value of $\|A\|^2$ should be $2n$. Indeed the immersion $f\colon SO(3) \to S^4$ of the Lie group $SO(3)$ defined by $f(g) = gBg^{-1}$, where B is a 3×3 diagonal matrix with diagonal $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0$ is a minimal immersion with $\|A\|^2 = 6$ (cf. [6]). Then Yang and Cheng (cf. [13, 14]) improved the result of Peng and Terng by proving $c(n) > \frac{2}{7}n - \frac{9}{14}$. These authors in [12] further improved this result by proving if $\|A\|^2 > n$, then $\|A\|^2 \geq \frac{1}{3}(4n+1)$. In this paper we prove the following Theorem.

Theorem 2. *Let M be a compact minimal hypersurface of constant scalar curvature in the unit sphere S^{2n+1}. If the shape operator A and the Ricci curvature of M satisfy $\|A\|^2 > 2n$, and $\mathrm{Ric} < 2(n-1)$, then there exists an eigenvalue $\lambda > 4n$ of the Laplace operator on M satisfying $\|A\|^2 = \lambda - 2n$.*

Other important question in the geometry of compact minimal hypersurface in the unit sphere S^{n+1} is to show that the first nonzero eigenvalue λ_1 of its Laplacian operator satisfies $\lambda_1 = n$, known as Yau's problem (cf. [15]). For embedded compact minimal hypersurfaces it has been known that $\lambda_1 \geq \frac{n}{2}$ (cf. [5]), however no such result is available for immersed minimal hypersurfaces in S^{n+1}. In this paper we prove the following result for an immersed compact minimal Einstein hypersurface of the unit sphere S^{n+1}:

Theorem 3. *Let M be an immersed compact minimal Einstein hypersurface of the unit sphere S^{n+1}, $n>2$. Then the first nonzero eigenvalue λ_1 of the Laplacian operator on M satisfies*

$$\lambda_1 \geq n\left(1 - \frac{1}{n-1}\right)$$

2. Preliminaries

Let M be an immersed compact minimal hypersurface of the unit sphere S^{n+1} with unit normal vector field N and shape operator A. We denote by ∇ and $\overline{\nabla}$ the Riemannian connections on M and S^{n+1} respectively and by g the

Riemannian metric on S^{n+1} as well as that induced on M. The Ricci tensor Ric and the scalar curvature S of M are given by (cf. [2])

$$\operatorname{Ric}(X,Y) = (n-1)g(X,Y) - g(AX,AY), \quad S = n(n-1) - \|A\|^2 \tag{2.1}$$

$X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie-algebra of smooth vector fields on M. For a constant vector field $\mathfrak{a}$ on R^{n+2}, we define smooth functions $f, h\colon M \to R$ by

$$f = \langle \mathfrak{a}, N\rangle, \quad h = \langle \mathfrak{a}, \overline{N}\rangle \tag{2.2}$$

where $\langle , \rangle$ is the Euclidean metric on R^{n+2} and consequently the restriction of $\mathfrak{a}$ to M can be expressed as

$$\mathfrak{a} = t + fN + h\overline{N} \tag{2.3}$$

where $t \in \mathfrak{X}(M)$ is the tangential component of $\mathfrak{a}$ to M. Using Gauss formula for the hypersurface M in S^{n+1} and for the hypersurface S^{n+1} in R^{n+2}, we obtain

$$\nabla_X t = fA(X) - hX, \quad X(f) = -g(At, X), \quad X(h) = g(t, X) \tag{2.4}$$

$X \in \mathfrak{X}(M)$, and consequently the gradient fields ∇f, ∇h of the functions f, h are given by

$$\nabla f = -A(t), \quad \nabla h = t \tag{2.5}$$

Since M is minimal hypersurface, using equations (2.4) and (2.5), we obtain the following expressions for the Laplacians Δf and Δh of the functions f and h

$$\Delta f = -\|A\|^2 f, \quad \Delta h = -nh \tag{2.6}$$

Using the fact $\frac{1}{2}\Delta f^2 = f\Delta f + \|\nabla f\|^2$ and the equations (2.5) and (2.6) we have the following

Lemma 2.1. *Let M be a compact orientable minimal hypersurface of the unit sphere S^{n+1}. Then*

$$\int_M \|t\|^2 = n\int_M h^2, \quad \int_M \|A(t)\|^2 = \int_M \|A\|^2 f^2.$$

An odd dimensional unit sphere S^{2n+1} in the Euclidean space R^{2n+2} inherits contact structure induced by the complex structure J on R^{2n+2}. The unit normal vector field $\overline{N}$ of the unit sphere defines a unit vector field $\xi = -J\overline{N}$ on the sphere S^{2n+1} with its dual form η and a tensor filed φ of type (1,1) defined by

$$\overline{\nabla}_X \xi = -\varphi X \tag{2.7}$$

for a smooth vector field X on S^{2n+1}. This gives contact structure (φ, ξ, η, g) on the unit sphere S^{2n+1} that satisfies (cf. [1])

$$\varphi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \varphi\xi = 0, \ \eta(\varphi X) = 0,$$

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y)$$
$$\eta(X)=g(X,\xi),\ \left(\overline{\nabla}_X\varphi\right)(Y)=g(X,Y)\xi-\eta(Y)X$$

for smooth vector fields X,Y on S^{2n+1}. For an immersed hypersurface M of the unit sphere S^{2n+1} with unit normal vector field N, $\varphi(N)$ is tangential to M and thus we put $u=-\varphi(N)$ where $u\in\mathfrak{X}(M)$. Define a smooth function $\rho=g(\xi,N)$ on M and thus we express the restrictions of ξ and φX to M, $X\in\mathfrak{X}(M)$ as

$$\xi=v+\rho N,\quad \varphi X=\psi X+\alpha(X)N \tag{2.8}$$

where v, $\psi(X)$ are tangential components of ξ and φX to M respectively and α is a 1-form on M dual to u, that is $\alpha(X)=g(X,u)$, $X\in\mathfrak{X}(M)$. Let β be the 1-form dual to the vector field v. Then the hypersurface M inherits the structure $(\psi,u,v,\alpha,\beta,g)$ which has the property summarized in the following Lemma the proof of which follows trivially by the properties of the contact structure on S^{2n+1} and the Gauss formula for the hypersurface.

Lemma 2.2. *Let M be an orientable hypersurface of the unit sphere S^{2n+1}. Then M inherits the structure $(\psi,u,v,\alpha,\beta,g)$ satisfying*

(i) $\psi^2X=-X+\alpha(X)u+\beta(X)v$, $\alpha(u)=\beta(v)=1-\rho^2$, $\psi(u)=-\rho v$, $\psi(v)=\rho u$, $\alpha(\psi X)=\rho\beta(X)$, $\beta(\psi X)=-\rho\alpha(X)$

(ii) $g(\psi X,\psi Y)=g(X,Y)-\alpha(X)\alpha(Y)-\beta(X)\beta(Y)$, $\alpha(X)=g(X,u)$, $\beta(X)=g(X,v)$, $g(\psi X,Y)=-g(X,\psi Y)$

(iii) $(\nabla_X\psi)(Y)=g(X,Y)v-\beta(Y)X+\alpha(Y)AX-g(AX,Y)u$, $\nabla_Xu=\rho X+\psi(AX)$, $\nabla_Xv=-\psi(X)+\rho AX$

where ∇ is the Riemannian connection on the hypersurface and $X,Y\in\mathfrak{X}(M)$.

For a non-totally geodesic compact minimal hypersurface M of constant scalar curvature in the unit sphere S^{n+1} by equations in (2.6) it follows that n and $\|A\|^2$ are eigenvalues of the Laplacian operator on M. It is an interesting question to see whether sum of two eigenvalues of Laplacian operator on a Riemannian manifold is also an eigenvalue of the Laplacian operator. Indeed for compact minimal hypersurface of constant scalar curvature in the odd dimensional unit sphere S^{2n+1}, $2n+\|A\|^2$ is also an eigenvalue of the Laplacian operator as seen in the following:

Lemma 2.3. *Let M be a compact minimal hypersurface of constant scalar curvature of the unit sphere S^{2n+1}. Then the function ρ satisfies*

$$\Delta\rho=-(2n+\|A\|^2)\rho$$

Proof. Using the definition of ρ and equations (2.7), (2.8) we immediately get the following expression for the gradient $\nabla\rho$

$$\nabla\rho=-u-Av \tag{2.9}$$

Now using (iii) in Lemma 2.2 and the skew-symmetry of the operator ψ, get

$$div(u)=2n\rho,\quad div(v)=\|A\|^2\rho$$

and consequently using this in equation (2.9) we have proved the Lemma. □

3. Proof of theorems

Proof of Theorem 1. Let M be the minimal hypersurface of the unit sphere S^{n+1} and $\mathfrak{a}$, be a nonzero constant vector field on R^{n+2} satisfying $\langle \mathfrak{a}, N\rangle = c\left\langle \mathfrak{a}, \overline{N}\right\rangle$ for a constant $c \neq 0$. Thus using $f = ch$ in equation (2.6) we conclude that $\left(n - \|A\|^2\right) h = 0$. Since M is connected, we have either $n = \|A\|^2$ or else $h = 0$. If $h = 0$, then by our assumption $f = 0$ and by first equation in Lemma 2.1 we have $t = 0$. This together with equation (2.3) and the fact that $\mathfrak{a}$ is a constant vector field implies that $\mathfrak{a} = 0$ which is a contradiction. Hence $\|A\|^2 = n$, $n > 2$ and this proves that M is a Clifford hypersurface $S^l\left(\sqrt{\frac{l}{n}}\right) \times S^m\left(\sqrt{\frac{m}{n}}\right)$, $l + m = n$ (cf. [3]).

Conversely suppose $M = S^l\left(\sqrt{\frac{l}{n}}\right) \times S^m\left(\sqrt{\frac{m}{n}}\right)$, $l+m = n$. Let $\Psi_1 \colon S^l\left(\sqrt{\frac{l}{n}}\right) \to R^{l+1}$ and $\Psi_2 \colon S^m\left(\sqrt{\frac{m}{n}}\right) \to R^{m+1}$ be the natural embeddings with unit normals N_1 and N_2 respectively. Then the embedding $\Psi = (\Psi_1, \Psi_2)$ gives the minimal hypersurface $M = S^l\left(\sqrt{\frac{l}{n}}\right) \times S^m\left(\sqrt{\frac{m}{n}}\right)$, $l + m = n$ of the unit sphere S^{n+1} and the unit normals N of M in S^{n+1} and $\overline{N}$ of S^{n+1} in R^{n+2} are given by

$$N = \left(\sqrt{\frac{m}{n}}N_1, -\sqrt{\frac{l}{n}}N_2\right), \quad \overline{N} = \left(\sqrt{\frac{l}{n}}N_1, \sqrt{\frac{m}{n}}N_2\right)$$

Then the coordinate vector field $\mathfrak{a} = \frac{\partial}{\partial x^1}$ on R^{n+2} satisfies $f = ch$, for the constant $c = \sqrt{\frac{m}{l}} \neq 0$. □

Proof of Theorem 2. Let M be the minimal hypersurface of the unit sphere S^{2n+1} with shape operator A and Ricci curvature satisfying the hypothesis of the Theorem. Then by Lemma 2.2, the function ρ satisfies

$$\Delta\rho = -(2n + \|A\|^2)\rho \tag{3.1}$$

We claim that the function ρ is not a constant on M. If is ρ a constant then by equation (3.1) we get $\rho = 0$ and consequently the equations (2.8) and (2.9) will imply that $\xi = v$ is tangent to M and that $A\xi = -u$, and that u is a unit vector field (by Lemma 2.2). Thus

$$\mathrm{Ric}(\xi, \xi) = (2n - 1) - 1 = 2(n - 1)$$

which is a contradiction. Hence ρ is a non-constant smooth function. Thus by equation (3.1) we see that ρ is an eigenfunction of the Laplacian operator corresponding to eigenvalue $\lambda = 2n + \|A\|^2 > 4n$, that is $\|A\|^2 = \lambda - 2n$. □

Proof of Theorem 3. Let M be a compact minimal Einstein hypersurface of the unit sphere S^{n+1}. Then its Ricci curvature tensor is given by

$$\mathrm{Ric} = \frac{S}{n}g$$

where S is the scalar curvature of M which is a constant as $n > 2$, and consequently $\|A\|^2$ is a constant. Moreover by equation (2.1) we have

$$A^2 = \frac{\|A\|^2}{n} I$$

This shows, as $trA = 0$ and eigenvalues of A are $\pm\frac{\|A\|}{\sqrt{n}}$, that $\dim M =$ even, say $2m$, and consequently M is a minimal hypersurface of the odd-dimensional unit sphere S^{2m+1} and therefore has $(\psi, u, v, \alpha, \beta, g)$-structure described in the Lemma 2.2.

Let M be a compact minimal Einstein hypersurface of the unit sphere S^{2m+1} and $\sigma\colon M \to R$ be a smooth function. For this smooth function we define an operator $B_\sigma\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$B_\sigma(X) = \nabla_X \nabla\sigma$$

Then the operator B_σ is symmetric and $trB_\sigma = \Delta\sigma$, moreover it is straightforward to verify that

$$(\nabla B_\sigma)(X, Y) - (\nabla B_\sigma)(Y, X) = R(X, Y)\nabla\sigma \tag{3.2}$$

where R is the curvature tensor field of the hypersurface and the covariant derivative $(\nabla B_\sigma)(X, Y) = \nabla_X B_\sigma(Y) - B_\sigma(\nabla_X Y)$. Also for a $X \in \mathfrak{X}(M)$ and a local orthonormal frame $\{e_1, \ldots, e_{2m}\}$ we have

$$X(\Delta\sigma) = X\left(\sum g(B_\sigma(e_i), e_i)\right) = \sum g((\nabla B_\sigma)(X, e_i), e_i)$$

which together with equation (3.2) gives

$$\sum_{i=1}^{2m} (\nabla B_\sigma)(e_i, e_i) = \nabla(\Delta\sigma) + \frac{S}{2m}\nabla\sigma \tag{3.3}$$

Now take σ as eigenfunction of Δ corresponding to first nonzero eigenvalue λ_1, that is $\Delta\sigma = -\lambda_1\sigma$. Then we have

$$\int_M \|\nabla\sigma\|^2 = \lambda_1 \int_M \sigma^2 \tag{3.4}$$

We use equation (3.3) to compute

$$\begin{aligned} div(B_\sigma(\nabla\sigma)) &= \|B_\sigma\|^2 + \sum g\left(\nabla\sigma, (\nabla B_\sigma)(e_i, e_i)\right) \\ &= \|B_\sigma\|^2 - \lambda_1 \|\nabla\sigma\|^2 + \mathrm{Ric}(\nabla\sigma, \nabla\sigma) \end{aligned} \tag{3.5}$$

If M is totally geodesic then we have $\lambda_1 = 2m = n$ and the result holds. Therefore suppose M is not totally geodesic. Then M is Clifford hypersurface (cf. [10]), and we have $\|A\|^2 = 2m$, consequently $A^2 = I$ which gives

$$\mathrm{Ric}(\nabla\sigma, \nabla\sigma) = 2(m-1)\|\nabla\sigma\|^2 \tag{3.6}$$

Thus integrating equation (3.5) and using (3.4) and (3-6) we get

$$\int\limits_M \left(\|B_\sigma\|^2 - \tfrac{\lambda_1^2}{2m}\sigma^2 \right) = \frac{\lambda_1}{2m} \left(\lambda_1(2m-1) - 4m(m-1) \right) \int\limits_M \sigma^2$$

As $trB_\sigma = -\lambda_1\sigma$, by Schwartz's inequality the first integrand in above equation is non-negative, which gives $\lambda_1(2m-1) \geq 4m(m-1)$ and this proves the Theorem. □

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Received August 9, 2008.

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