Q.1. Answer the following

(i) Consider the set

$$E = \{(x, y, z) \in R^3 : xy = 1, \quad 0 \leq z \leq 1\}.$$

Is the set E connected? For $p = (1, 1, 0) \in E$, find the component C_p. [2]

(ii) Let $X = D \cup \{(0, \frac{1}{2})\}$ be the subspace of R^2, where D is the deleted comb space. Find the path components of X. [3]

(iii) Give an example of a topological space X and a point $x \in X$ such that $C_x \neq Q_x$. [3]

(iv) Use the continuity of determinant function to check whether the general linear group $GL(n, R)$ is connected. [2]

Solution: (i) For the space $E = E_1 \cup E_2$, where

$$E_1 = \left\{(x, y, z) \in R^3 : xy = 1,\, x > 0 \quad 0 \leq z \leq 1\right\}$$

and

$$E_1 = \left\{(x, y, z) \in R^3 : xy = 1,\, x < 0 \quad 0 \leq z \leq 1\right\}$$

is a seperation and therefore E is not connected. Infact both E_1 and E_2 are connected and are components of E. $C_p = E_1$.

(ii) Let $p = (0, 1)$, $q = (0, \frac{1}{2})$ and $x \in D - \{p\}$. Then the path components of X are

$$C_p = \{(0, 1)\}, \quad C_q = \left\{(0, \frac{1}{2})\right\}, \quad C_x = D - \{p\}$$

(iii) Consider the subspace $X = \{(0, 0), (1, 0)\} \cup \left\{(x, y) \in R^2 : 0 \leq x \leq 1, y = \frac{1}{n},\, n \text{ a natural number}\right\}$ of R^2. If we denote $p = (0, 0)$ and $q = (1, 0)$. Then we get $C_p = \{p\}$ and $Q_p = \{p, q\}$ that is $C_p \neq Q_p$.

(iv) The function $f : GL(n, R) \to R$ defined by $f(X) = \det X$ is a continuous function and $f(GL(n, R)) = R - \{0\}$. If $GL(n, R)$ is connected, its continuous image $R - \{0\}$ is connected which is a contradiction. Hence $GL(n, R)$ is not connected.

Q.2. Answer the following

(i) Show that in a locally connected space X the components C_x are open subsets of X. Is the converse true? (Justify your answer by either giving a proof or a counter example). [3]

(ii) Show that the product $X \times Y$ is path connected if both X and Y are path connected. Is the converse true? [2]

(iii) If X is locally path connected space, then show that $P_x = C_x = Q_x$. [2]

(iv) Show that the subspaces $X = (0,1) \cup [2,3)$ and $Y = (0,1) \cup (2,3)$ of R are not homeomorphic. [3]

Solution: (i) Suppose X is locally connected and $y \in C_x$. Then for any open subset $U \subset X$ containing y, as X is locally connected, there is a connected neighbourhood V of y satisfying $y \in V \subset U$. However as C_x is the largest connected subset containing y, we get $y \in V \subset C_x$ and this proves C_x is open subset of X. The converse is not true for example the component $C_x = D$ of deleted comb space is open where as D is not locally connected. (Note: Do not get confused with the characterization of locally connected space where every open set is used that is X is locally connected if and only if for each open set U the components of U are open subsets of X)

(ii) Take two points $p = (x_1, y_1)$, $q = (x_2, y_2) \in X \times Y$. Then as X and Y are path connected, for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, there exists paths $\alpha : [0,1] \to X$ and $\beta : [0,1] \to Y$ such that $\alpha(0) = x_1, \alpha(1) = x_2$ and $\beta(0) = y_1, \beta(1) = y_2$ respectively. Then $\gamma : [0,1] \to X \times Y$ defined by $\gamma(t) = (\alpha(t), \beta(t))$ is the path in $X \times Y$ that joins p to q. This proves that $X \times Y$ is path connected. The converse is also true.

(iii) We know that in a locally path connected space X, we have $P_x = C_x$. Since a locally path connected space is locally connected we have $C_x = Q_x$. Thus in a locally path connected space we have $P_x = C_x = Q_x$.

(iv) If $X = (0,1) \cup [2,3)$ and $Y = (0,1) \cup (2,3)$ are homeomorphic, there is a hoemomorphism $f : X \to Y$ with $f(2) = a \in Y$. Thus $f : X - \{2\} \to Y - \{a\}$ is also a homeomorphism and which produces a one-to-one correspondence between the sets of components of $X - \{2\}$ and $Y - \{a\}$, which is a contradiction as $X - \{2\} = (0,1) \cup (2,3)$ has two components and $Y - \{a\}$ has three components. Hence X is not homeomorphic to Y.

Q.3. (i) Let $U \subset R^n$ be an open subset and $p \in U$. If $f : U \to R^m$ is differentiable at p, then show that there is a **unique** $T \in L(R^n, R^m)$ such that $D_p f = T$. [3]

(ii) Let $U \subset R^n$ be an open subset and $p \in U$. If $f : U \to R^m$ is differentiable at p, find the matrix of $D_p f$. [3]

(iii) Assume that the function $f : R^2 \to R^3$ defined by $f(x,y,z) = \left(\frac{1}{2}(x^2 - y^2), xy, \frac{1}{2}(x^2 + y^2)\right)$ is differentiable at $p = (1,1) \in R^2$ and find the rank of $D_p f$. [3]

Solution: (i) Suppose that there are two $T_1, T_2 \in L(R^n, R^m)$ satisfying the requirement of differentiability of f at $p \in U$, that is

$$\lim_{X \to 0} \frac{\|f(p+X) - f(p) - T_1(X)\|}{\|X\|} = 0, \ \lim_{X \to 0} \frac{\|f(p+X) - f(p) - T_2(X)\|}{\|X\|} = 0$$

This is equivalent to

$$f(p+X) = f(p) + T_1(X) + \|X\| \, \epsilon_1(X), \ f(p+X) = f(p) + T_2(X) + \|X\| \, \epsilon_2(X)$$

where $\epsilon_i(X) \to 0$ as $X \to 0$ for $i = 1, 2$.

Thus we have for $t \in R$ and using above relations that

$$\begin{aligned} \|T_1(X) - T_2(X)\| &= \lim_{t\to 0}\frac{|t|}{|t|}\|T_1(X) - T_2(X)\| = \lim_{t\to 0}\frac{1}{|t|}\|T_1(tX) - T_2(tX)\| \\ &= \lim_{t\to 0}\frac{1}{|t|}\|f(p+tX) - f(p) - \|tX\|\epsilon_1(tX) - f(p+tX) + f(p) + \|tX\|\epsilon_2(tX)\| \\ &= \lim_{t\to 0}\frac{\|tX\|}{|t|}\|-\epsilon_1(tX) + \epsilon_2(tX)\| = \lim_{t\to 0}\|X\|\,\|-\epsilon_1(tX) + \epsilon_2(tX)\| = 0 \end{aligned}$$

Hence $T_1 = T_2$.

(ii) Now suppose $f : U \to R^m$, ($U \subset R^n$ is open subset) is differentiable at $p \in U$. Then as the derivative $D_pf : R^n \to R^m$ is a linear transformation, we choose the canonical bases $\{e_1, .., e_n\} \subset R^n$ and $\{E_1, .., E_m\} \subset R^m$, that is $e_i = (0, .., 0, 1, 0, .., 0) \in R^n$ that is a vector in R^m with 1 at ith place and 0 at other places, similarly $E_i = (0, .., 0, 1, 0, , 0) \in R^m$. In order to find the matrix of D_pf, we need to find the value of $(D_pf)(e_j)$ and express it as linear combination of the vectors in the basis $\{E_1, .., E_m\}$. We observe that any $u = (\lambda_1, .., \lambda_m) \in R^m$ is expressed as

$$u = \lambda_1 E_1 + .. + \lambda_m E_m = \sum_{i=1}^{m}\lambda_i E_i$$

Using equation (6) we have

$$(D_pf)(e_j) = \left(\left(D_pf^1\right)(e_j), .., (D_pf^m)(e_j)\right) \in R^m$$

and consequently

$$(D_pf)(e_j) = \sum_{i=1}^{m}\left(D_pf^i\right)(e_j)E_i \tag{7}$$

Now as $f^i : U \to R$ is differentiable at p, we can use Proposition-2 to evaluate $\left(D_pf^i\right)(e_j)$ as

$$\left(D_pf^i\right)(e_j) = \lim_{t\to 0}\frac{1}{t}\left[f^i(p+te_j) - f^i(p)\right]$$

Note that if $p = (a_1, .., a_j, .., a_n)$, then $p + te_j = (a_1, .., a_j + t, .., a_n)$ and consequently

$$\begin{aligned} \lim_{t\to 0}\frac{1}{t}\left[f^i(p+te_j) - f(p)\right] &= \lim_{t\to 0}\frac{1}{t}\left[f^i((a_1, .., a_j + t, .., a_n)) - f^i((a_1, .., a_j, .., a_n))\right] \\ &= \frac{\partial f^i}{\partial x^j}(p) \end{aligned}$$

where $\frac{\partial f^i}{\partial x^j}(p)$ is the partial derivative of f^i with respect to the jth coordinate at the point p. Thus we have computed

$$\left(D_pf^i\right)(e_j) = \frac{\partial f^i}{\partial x^j}(p)$$

and then by equation (7) we have

$$(D_pf)(e_j) = \sum_{i=1}^{m}\frac{\partial f^i}{\partial x^j}(p)E_i$$

This gives the $m \times n$ matrix of the derivative D_pf as

$$D_pf = \begin{bmatrix} \frac{\partial f^1}{\partial x^1}(p) & .. & .. & \frac{\partial f^1}{\partial x^n}(p) \\ . & .. & .. & . \\ . & .. & .. & . \\ \frac{\partial f^m}{\partial x^1}(p) & .. & .. & \frac{\partial f^m}{\partial x^n}(p) \end{bmatrix}$$

(iii) We have $f^1 = \frac{1}{2}(x^2 - y^2)$, $f^2 = xy$ and $f^3 = \frac{1}{2}(x^2 + y^2)$, consequently at the point $p = (1,1)$

$$D_p f = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Since the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is non-singular, $Rank D_p f = 2$.

Q.4. (i) Show that the function $f : R^2 \to R$ defined by

$$f(x,y) = e^{2x} \cos x$$

is differentiable at $p = (a,b) \in R^2$ and find $D_p f$. [4]

(ii) Let $U, V \subset R^n$ be an open subsets and $p \in U$. If $f : U \to V$ is one-one on-to differentiable at p and $f^{-1} : V \to U$ is differentiable at $q = f(p)$, then show that $D_p f$ is non-singular and find $(D_p f)^{-1}$. [3]

(iii) Assume that the function $f : R^2 \to R^2$ defined by $f(x,y) = \left(\frac{1}{2}(x^2 - y^2), \frac{1}{2}(x^2 + y^2)\right)$ is differentiable at $p = (1,1) \in R^2$ and show that there exist neighbourhoods U and V of p and $q = (0,1)$ respectively, such that $f : U \to V$ is one-one and on-to. Is $f : R^2 \to R^2$ one-one on-to? [3]

Solution: (i) Consider the map $T : R^2 \to R$ defined by $T(x,y) = xe^{2a}(2\cos a - \sin a)$ and it is easy to show that T is a linear transformation and thus $T \in L(R^2, R)$. Using L'Hospitals rule one can show that

$$\lim_{(x,y)\to(0,0)} \frac{|f(p+X) - f(p) - T(X)|}{\|X\|} = 0$$

where $X = (x,y)$. Hence f is differentiable at p and $D_p f = T$.

(ii) We have $f \circ f^{-1} = I$ and $f^{-1} \circ f = I$. Then letting $f(p) = q$ and using Chain rule we have $D_p\left(f^{-1} \circ f\right) = I$ and $D_q\left(f \circ f^{-1}\right) = I$ which gives

$$D_q f^{-1} \circ D_p f = I \text{ and } D_p f \circ D_q f^{-1} = I$$

that is the linear transformation $D_p f$ is invertible and its inverse is given by

$$(D_p f)^{-1} = D_q f^{-1}$$

In particular we see that $\det(D_p f) \neq 0$ at each point p.

(iii) We have f differentiable at $p = (1,1)$ and $f^1 = \frac{1}{2}(x^2 - y^2)$ and $f^2 = \frac{1}{2}(x^2 + y^2)$ and thus the matrix of $D_p f$ is given by

$$D_p f = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

We have $\det D_p f = 2 \neq 0$ and thus by inverse function theorem there exist neighbourhoods U and V of p and $f(p) = (0,1) = q$ respectively such that $f : U \to V$ is one-one and on-to. Note that $f(1,1) = f(-1,-1)$ and therefore, f is not one-one. Also, for $(0,-1) \in R^2$ there is no point in R^2 such that $f(x,y) = (0,-1)$ and thus f is not on-to.

Q.5. (i) Let $U \subset S^2$ be defined by $U = \{(x,y,z) \in S^2 : x > 0\}$ and $B_1^2(0) \subset R^2$ be the open ball in R^2 of radius 1 centered at origin. Show that (U, φ) is a chart on S^2, where $\varphi : U \to B_1^2(0)$ is defined by

$$\varphi(x,y,z) = (y,z)$$

Also show that this chart belongs to the differentiable structure of S^2. [4]

(ii) Let $f : S^2 \to R$ be $f(x,y,z) = z$, and $p = (0,0,1). \in S^2$. Show that f is smooth at p. Choose a chart (U, φ) around p on S^2 with local coordinates x^1, x^2 and find the partial derivatives

$$\frac{\partial f}{\partial x^1}(p), \quad \frac{\partial f}{\partial x^2}(p)$$

[4]

(iii) Consider the functions $f, g : R^2 \to R$ defined by $f(x,y) = \frac{1}{2}(x^2 - y^2)$, $g(x,y) = \frac{1}{2}(x^2 + y^2)$. Show that there exists a chart (U, φ) around $p = (1,1) \in R^2$ with local coordinates f, g. [3]

Solution: (i) First note that as $\pi_1 : R^3 \to R$, $\pi(x,y,z) = x$ is a continuous function the set $\pi_1^{-1}(0,\infty) \cap S^2 = U$ is open subset of S^2. Now as the component functions of φ are continuous, the map $\varphi : U \to B_1^2(0)$ is continuous. Moreover, if $\varphi(x,y,z) = \varphi(a,b,c)$, (x,y,z), $(a,b,c) \in U$. Then we have $(y,z) = (b,c)$, that is $y = b$ and $z = c$. Thus we have $x^2 = 1 - y^2 - z^2 = 1 - b^2 - c^2 = a^2$ that is $x = \pm a$. As both x and a are positive, we have $x = a$ and this proves that φ is one-one. Also for $(u,v) \in B_1^2(0)$, as $u^2 + v^2 < 1$, we get $\left(\sqrt{1 - u^2 - v^2}, u, v\right) \in U$ and that

$$\varphi\left(\sqrt{1 - u^2 - v^2}, u, v\right) = (u,v)$$

that is φ is on-to. The inverse of φ is given by

$$\varphi^{-1}(u,v) = \left(\sqrt{1 - u^2 - v^2}, u, v\right)$$

which is also continuous and consequently φ is a homeomorphism and this proves that (U, φ) is a chart on S^2. We know that the differentiable structure S on S^2 is given by the collection $\{(V, \psi), (W, f)\}$ where $V = S^2 - \{(0,0,1)\}$ and $W = S^2 - \{(0,0,-1)\}$ and the maps ψ, f are defined by

$$\psi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right), \; f(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

with their inverses

$$\psi^{-1}(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right)$$

and

$$f^{-1}(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2}\right)$$

Observe that $U \cap V \neq \phi$ and $U \cap W \neq \phi$ and we have to check for the smooth ness of the maps $\varphi \circ \psi^{-1}$, $\psi \circ \varphi^{-1}$, $\varphi \circ f^{-1}$ and $f \circ \varphi^{-1}$. We have

$$\begin{aligned} \varphi \circ \psi^{-1}(u,v) &= \varphi\left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right) \\ &= \left(\frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right) \end{aligned}$$

which is clearly a smooth map. Similary we can show for other three maps $\psi \circ \varphi^{-1}$, $\varphi \circ f^{-1}$ and $f \circ \varphi^{-1}$. Hence $(U, \varphi) \in S$.

(ii) We choose a chart (U, φ) on S^2 defined by $U = \{(x, y, z) \in S^2 : z > 0\}$ and $\varphi : U \to B_1^2(0)$ given by $\varphi(x, y, z) = (x, y)$ where $B_1^2(0)$ is the open ball of radius 1 centered at origin in R^2. It is easy to show that this chart is in the differentialbe structure of S^2 with

$$\varphi^{-1}(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right)$$

We have

$$f \circ \varphi^{-1}(u, v) = f\left(u, v, \sqrt{1 - u^2 - v^2}\right) = \sqrt{1 - u^2 - v^2}$$

which is smooth and therefore the function f is smooth. Now we compute the partial derivatives of f with respect to the local coordinates x^1, x^2 on U. We have (note that $\varphi(p) = (0, 0)$)

$$\begin{aligned} \frac{\partial f}{\partial x^1}(p) &= \frac{\partial}{\partial u}(f \circ \varphi^{-1})(0, 0) \\ &= \frac{\partial}{\partial u} f\left(u, v, \sqrt{1 - u^2 - v^2}\right)(0, 0) \\ &= \frac{\partial}{\partial u}\left(\sqrt{1 - u^2 - v^2}\right)(0, 0) = 0 \end{aligned}$$

Similarly we compute

$$\frac{\partial f}{\partial x^2}(p) = 0$$

(iii) On R^2 we have the local coordinates x, y defined on R^2 and we have the functioinal determinant

$$\frac{D(f, g)}{D(x, y)}(p) = \det \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = 2 \neq 0$$

Hence there exists a chart (U, φ) around p on R^2 with local coordinates f, g.