Chapter 4: Mathematical Expectation:

4.1 Mean of a Random Variable:

Definition 4.1:

Let X be a random variable with a probability distribution f(x). The mean (or expected value) of X is denoted by μ_X (or E(X)) and is defined by:

$$E(X)=\mu_X=\begin{cases}\sum_{all\ x} x\, f(x)\,; & if\ \ X\ \ is\ discrete\\ \int_{-\infty}^{\infty} x\, f(x)\, dx\,; & if\ \ X\ \ is\ continuous\end{cases}$$

Example 4.1: (Reading Assignment)

Example: (Example 3.3)

A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective and 5 are non-defective. If a school makes a random purchase of 2 of these computers, find the expected number of defective computers purchased

Solution:

Let X = the number of defective computers purchased.

In Example 3.3, we found that the probability distribution of X is:

x	0	1	2
f(x)= P(X=x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

or:

$$f(x)=P(X=x)=\begin{cases}\dfrac{\binom{3}{x}\times\binom{5}{2-x}}{\binom{8}{2}}; & x=0,1,2\\ 0; \ otherwise\end{cases}$$

The expected value of the number of defective computers purchased is the mean (or the expected value) of X, which is:

$$E(X)=\mu_X=\sum_{x=0}^{2} x\, f(x)$$

$$= (0)\ f(0) + (1)\ f(1) +(2)\ f(2)$$

$$= (0)\,\frac{10}{28} + (1)\,\frac{15}{28} + (2)\,\frac{3}{28}$$

$$= \frac{15}{28} + \frac{6}{28} = \frac{21}{28} = 0.75 \text{ (computers)}$$

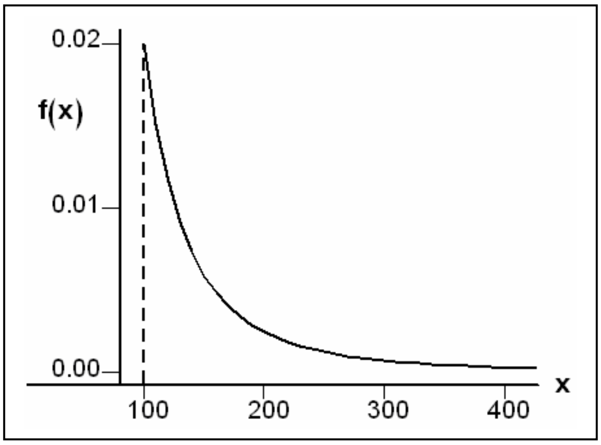
Example 4.3:

Let X be a continuous random variable that represents the life (in hours) of a certain electronic device. The pdf of X is given by:

$$f(x) = \begin{cases} \dfrac{20{,}000}{x^3} \;;\; x > 100 \\ 0 \;;\; elsewhere \end{cases}$$

Find the expected life of this type of devices.

Solution:



$$E(X) = \mu_X = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{100}^{\infty} x\, \frac{20000}{x^3}\, dx$$

$$= 20000 \int_{100}^{\infty} \frac{1}{x^2}\, dx$$

$$= 20000 \left[-\frac{1}{x} \Big|_{x=100}^{x=\infty} \right]$$

$$= -20000 \left[0 - \frac{1}{100} \right] = 200 \text{ (hours)}$$

Therefore, we expect that this type of electronic devices to last, on average, 200 hours.

Theorem 4.1:

Let X be a random variable with a probability distribution f(x), and let g(X) be a function of the random variable X. The mean (or expected value) of the random variable g(X) is denoted by $\mu_{g(X)}$ (or E[g(X)]) and is defined by:

$$E[g(X)] = \mu_{g(X)} = \begin{cases} \sum_{all\ x} g(x)\, f(x) \;; & if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} g(x)\, f(x)\, dx \;; & if\ X\ is\ continuous \end{cases}$$

Example:

Let X be a discrete random variable with the following probability distribution

x	0	1	2
f(x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

Find E[g(X)], where $g(X)=(X-1)^2$.

Solution:

$g(X)=(X-1)^2$

$$E[g(X)]=\mu_{g(X)}=\sum_{x=0}^{2} g(x)\, f(x)=\sum_{x=0}^{2} (x-1)^2\, f(x)$$

$$=(0-1)^2\, f(0)+(1-1)^2\, f(1)+(2-1)^2\, f(2)$$

$$=(-1)^2\,\frac{10}{28}+(0)^2\,\frac{15}{28}+(1)^2\,\frac{3}{28}$$

$$=\frac{10}{28}+0+\frac{3}{28}=\frac{13}{28}$$

Example:

In Example 4.3, find $E\left(\frac{1}{X}\right)$. {note: $g(X)=\frac{1}{X}$}

Solution:

$$f(x)=\begin{cases}\frac{20{,}000}{x^3} ; & x>100\\ 0 ; & elsewhere\end{cases}$$

$$g(X)=\frac{1}{X}$$

$$E\left(\frac{1}{X}\right)=E[g(X)]=\mu_{g(X)}=\int_{-\infty}^{\infty} g(x)\, f(x)\,dx=\int_{-\infty}^{\infty}\frac{1}{x}\, f(x)\,dx$$

$$=\int_{100}^{\infty}\frac{1}{x}\frac{20000}{x^3}\,dx=20000\int_{100}^{\infty}\frac{1}{x^4}\,dx=\frac{20000}{-3}\left[\frac{1}{x^3}\Bigg|_{x=100}^{x=\infty}\right]$$

$$=\frac{-20000}{3}\left[0-\frac{1}{1000000}\right]=0.0067$$

4.2 Variance (of a Random Variable):

The most important measure of variability of a random variable X is called the variance of X and is denoted by Var(X) or σ_X^2.

Definition 4.3:

Let X be a random variable with a probability distribution f(x) and mean μ. The variance of X is defined by:

$$\text{Var}(X)=\sigma_X^2=E[(X-\mu)^2]=\begin{cases}\sum_{all\ x}(x-\mu)^2 f(x)\,; & if\ X\ is\ discrete\\ \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx\,; & if\ X\ is\ continuous\end{cases}$$

Definition:

The positive square root of the variance of X, $\sigma_X=\sqrt{\sigma_X^2}$, is called the standard deviation of X.

Note:

Var(X)=E[g(X)], where $g(X)=(X-\mu)^2$

Theorem 4.2:

The variance of the random variable X is given by:

$$\text{Var}(X)=\sigma_X^2=E(X^2)-\mu^2$$

where $E(X^2)=\begin{cases}\sum_{all\ x}x^2 f(x)\,; & if\ X\ is\ discrete\\ \int_{-\infty}^{\infty}x^2 f(x)dx\,; & if\ X\ is\ continuous\end{cases}$

Example 4.9:

Let X be a discrete random variable with the following probability distribution

x	0	1	2	3
f(x)	0.51	0.38	0.10	0.01

Find Var(X)=σ_X^2.

Solution:

$\mu=\sum_{x=0}^{3}x\,f(x)$ = (0) f(0) + (1) f(1) +(2) f(2) + (3) f(3)

= (0) (0.51) + (1) (0.38) +(2) (0.10) + (3) (0.01)

= 0.61

1. First method:

$$\text{Var(X)}=\sigma_X^2=\sum_{x=0}^{3}(x-\mu)^2\ f(x)$$

$$=\sum_{x=0}^{3}(x-0.61)^2\ f(x)$$

$$=(0-0.61)^2\ f(0)+(1-0.61)^2\ f(1)+(2-0.61)^2\ f(2)+(3-0.61)^2\ f(3)$$

$$=(-0.61)^2\ (0.51)+(0.39)^2\ (0.38)+(1.39)^2\ (0.10)+(2.39)^2\ (0.01)$$

$$=0.4979$$

2. Second method:

$$\text{Var(X)}=\sigma_X^2=E(X^2)-\mu^2$$

$$E(X^2)=\sum_{x=0}^{3}x^2\ f(x)=(0^2)\ f(0)+(1^2)\ f(1)+(2^2)\ f(2)+(3^2)\ f(3)$$

$$=(0)\ (0.51)+(1)\ (0.38)+(4)\ (0.10)+(9)\ (0.01)$$

$$=0.87$$

$$\text{Var(X)}=\sigma_X^2=E(X^2)-\mu^2=0.87-(0.61)^2=0.4979$$

Example 4.10:

Let X be a continuous random variable with the following pdf:

$$f(x)=\begin{cases}2(x-1)\ ;\ 1<x<2\\ 0\ ;\ elsewhere\end{cases}$$

Find the mean and the variance of X.

Solution:

$$\mu=E(X)=\int_{-\infty}^{\infty}x\,f(x)\,dx=\int_{1}^{2}x[2(x-1)]\,dx=2\int_{1}^{2}x(x-1)\,dx=5/3$$

$$E(X^2)=\int_{-\infty}^{\infty}x^2\,f(x)\,dx=\int_{1}^{2}x^2[2(x-1)]\,dx=2\int_{1}^{2}x^2(x-1)\,dx=17/6$$

$$\text{Var(X)}=\sigma_X^2=E(X^2)-\mu^2=17/6-(5/3)^2=1/18$$

4.3 Means and Variances of Linear Combinations of Random Variables:

If $X_1, X_2, \ldots, X_n$ are n random variables and $a_1, a_2, \ldots, a_n$ are constants, then the random variable :

$$Y = \sum_{i=1}^{n} a_i X_i = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$$

is called a linear combination of the random variables $X_1, X_2, \ldots, X_n$.

Theorem 4.5:

If X is a random variable with mean μ=E(X), and if a and b are constants, then:

$$E(aX \pm b) = a\,E(X) \pm b$$

$$\Leftrightarrow$$

$$\mu_{aX\pm b} = a\,\mu_X \pm b$$

Corollary 1: E(b) = b (a=0 in Theorem 4.5)

Corollary 2: E(aX) = a E(X) (b=0 in Theorem 4.5)

Example 4.16:

Let X be a random variable with the following probability density function:

$$f(x) = \begin{cases} \frac{1}{3}x^2 \ ; -1 < x < 2 \\ 0 \ ; \ elsewhere \end{cases}$$

Find E(4X+3).

Solution:

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-1}^{2} x[\frac{1}{3}x^2]\, dx = \frac{1}{3}\int_{-1}^{2} x^3\, dx = \frac{1}{3}\left[\frac{1}{4}x^4 \Big|_{x=-1}^{x=2}\right] = 5/4$$

E(4X+3) = 4 E(X)+3 = 4(5/4) + 3=8

Another solution:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx \qquad ; g(X) = 4X+3$$

$$E(4X+3) = \int_{-\infty}^{\infty} (4x+3)\, f(x)\, dx = \int_{-1}^{2} (4x+3)\, [\frac{1}{3}x^2]\, dx = \cdots = 8$$

Theorem:

If $X_1, X_2, \ldots, X_n$ are n random variables and $a_1, a_2, \ldots, a_n$ are constants, then:

$$E(a_1X_1+a_2X_2+ \ldots +a_nX_n) = a_1E(X_1)+ a_2E(X_2)+ \ldots+a_nE(X_n)$$

$$\Leftrightarrow$$

$$E(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i E(X_i)$$

Corollary:

If X, and Y are random variables, then:

$$E(X \pm Y) = E(X) \pm E(Y)$$

Theorem 4.9:

If X is a random variable with variance $Var(X) = \sigma_X^2$ and if a and b are constants, then:

$$Var(aX \pm b) = a^2 \, Var(X)$$

$$\Leftrightarrow$$

$$\sigma_{aX+b}^2 = a^2 \sigma_X^2$$

Theorem:

If $X_1, X_2, \ldots, X_n$ are n <u>independent</u> random variables and $a_1, a_2, \ldots, a_n$ are constants, then:

$$Var(a_1X_1+a_2X_2+\ldots+a_nX_n)$$
$$= a_1^2 Var(X_1)+ a_2^2 Var(X_2)+\ldots+ a_n^2 Var(X_n)$$

$$\Leftrightarrow$$

$$Var(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i^2 Var(X_i)$$

$$\Leftrightarrow$$

$$\sigma_{a_1X_1+a_2X_2+\ldots+a_nX_n}^2 = a_1^2\sigma_{X_1}^2 + a_2^2\sigma_{X_2}^2 + \ldots + a_n^2\sigma_{X_n}^2$$

Corollary:

If X, and Y are <u>independent</u> random variables, then:

- $Var(aX+bY) = a^2 \, Var(X) + b^2 \, Var(Y)$
- $Var(aX-bY) = a^2 \, Var(X) + b^2 \, Var(Y)$
- $Var(X \pm Y) = Var(X) + Var(Y)$

Example:

Let X, and Y be two independent random variables such that E(X)=2, Var(X)=4, E(Y)=7, and Var(Y)=1. Find:

1. E(3X+7) and Var(3X+7)
2. E(5X+2Y−2) and Var(5X+2Y−2).

Solution:

1. $E(3X+7) = 3E(X)+7 = 3(2)+7 = 13$

   $Var(3X+7)= (3)^2\, Var(X)=(3)^2\,(4) =36$

2. $E(5X+2Y-2)= 5E(X) + 2E(Y) -2= (5)(2) + (2)(7) - 2= 22$

   $Var(5X+2Y-2)= Var(5X+2Y)= 5^2\, Var(X) + 2^2\, Var(Y)$

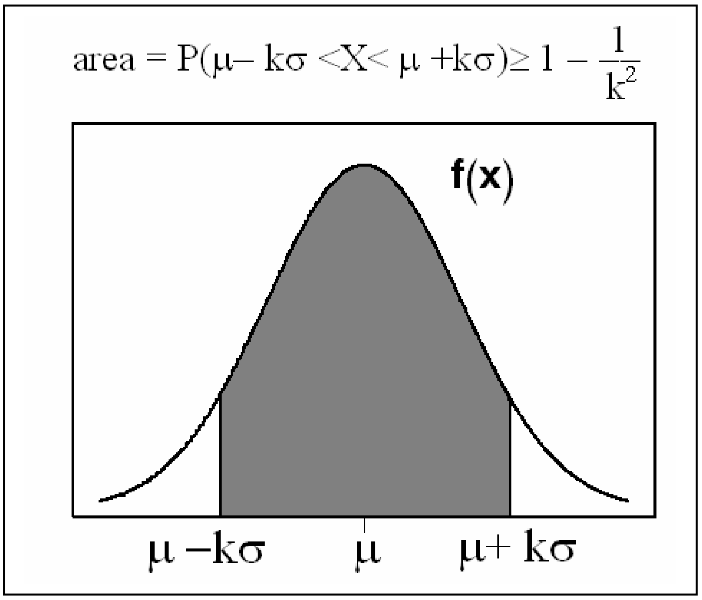
   $= (25)(4)+(4)(1) = 104$

4.4 Chebyshev's Theorem:

* Suppose that X is any random variable with mean $E(X)=\mu$ and variance $Var(X)=\sigma^2$ and standard deviation σ.

* Chebyshev's Theorem gives a conservative estimate of the probability that the random variable X assumes a value within k standard deviations ($k\sigma$) of its mean μ, which is $P(\mu-k\sigma<X<\mu+k\sigma)$.

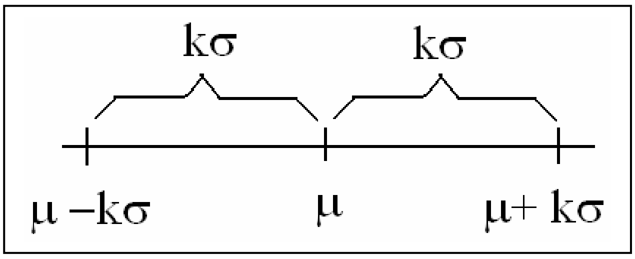
* $P(\mu-k\sigma<X<\mu+k\sigma)\approx 1-\frac{1}{k^2}$



Theorem 4.11:(Chebyshev's Theorem)

Let X be a random variable with mean $E(X)=\mu$ and variance $Var(X)=\sigma^2$, then for k>1, we have:

$$P(\mu-k\sigma<X<\mu+k\sigma)\geq 1-\frac{1}{k^2} \quad \Leftrightarrow \quad P(|X-\mu|<k\sigma)\geq 1-\frac{1}{k^2}$$



Example 4.22:

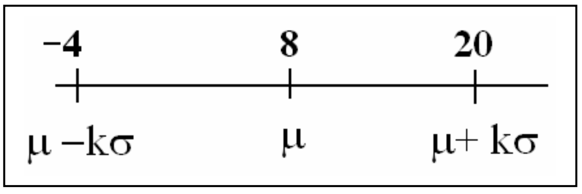
Let X be a random variable having an unknown distribution with mean $\mu=8$ and variance $\sigma^2=9$ (standard deviation $\sigma=3$). Find the following probability:

(a) $P(-4<X<20)$

(b) $P(|X-8|\geq 6)$

Solution:

(a) $P(-4<X<20)=$??



$$P(\mu-k\sigma<X<\mu+k\sigma)\geq 1-\frac{1}{k^2}$$

$(-4 < X < 20) = (\mu - k\sigma < X < \mu + k\sigma)$

$-4 = \mu - k\sigma \Leftrightarrow -4 = 8 - k(3)$ or $20 = \mu + k\sigma \Leftrightarrow 20 = 8 + k(3)$

$\Leftrightarrow -4 = 8 - 3k$ $\qquad \Leftrightarrow 20 = 8 + 3k$

$\Leftrightarrow 3k = 12$ $\qquad \Leftrightarrow 3k = 12$

$\Leftrightarrow$ **k=4** $\qquad \Leftrightarrow$ **k=4**

$$1 - \frac{1}{k^2} = 1 - \frac{1}{16} = \frac{15}{16}$$

Therefore, $P(-4 < X < 20) \geq \frac{15}{16}$, and hence, $P(-4 < X < 20) \approx \frac{15}{16}$ (approximately)

(b) $P(|X - 8| \geq 6) = ??$

$P(|X-8| \geq 6) = 1 - P(|X-8| < 6)$

$P(|X - 8| < 6) = ??$

$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$

$(|X-8| < 6) = (|X - \mu| < k\sigma)$

$6 = k\sigma \Leftrightarrow 6 = 3k \Leftrightarrow$ **k=2**

$$1 - \frac{1}{k^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

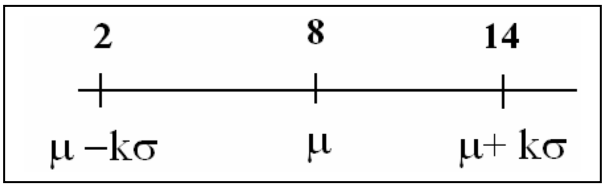
$P(|X-8| < 6) \geq \frac{3}{4} \Leftrightarrow 1 - P(|X-8| < 6) \leq 1 - \frac{3}{4}$

$\Leftrightarrow 1 - P(|X-8| < 6) \leq \frac{1}{4}$

$\Leftrightarrow P(|X-8| \geq 6) \leq \frac{1}{4}$

Therefore, $P(|X-8| \geq 6) \approx \frac{1}{4}$ (approximately)

Another solution for part (b):



$P(|X-8| < 6) = P(-6 < X-8 < 6)$

$= P(-6 + 8 < X < 6 + 8)$

$= P(2 < X < 14)$

$(2 < X < 14) = (\mu - k\sigma < X < \mu + k\sigma)$

$2 = \mu - k\sigma \Leftrightarrow 2 = 8 - k(3) \Leftrightarrow 2 = 8 - 3k \Leftrightarrow 3k = 6 \Leftrightarrow$ **k=2**

$$1-\frac{1}{k^2}=1-\frac{1}{4}=\frac{3}{4}$$

$$P(2<X<14) \ge \frac{3}{4} \Leftrightarrow P(|X-8|<6) \ge \frac{3}{4}$$

$$\Leftrightarrow 1 - P(|X-8|<6) \le 1 - \frac{3}{4}$$

$$\Leftrightarrow 1 - P(|X-8|<6) \le \frac{1}{4}$$

$$\Leftrightarrow P(|X-8| \ge 6) \le \frac{1}{4}$$

Therefore, $P(|X-8| \ge 6) \approx \frac{1}{4}$ (approximately)