Chapter 2: Probability:

- **An Experiment:** is some procedure (or process) that we do and it results in an outcome.

2.1 The Sample Space:

Definition 2.1:

- The set of all possible outcomes of a statistical experiment is called the sample space and is denoted by S.
- Each outcome (element or member) of the sample space S is called a sample point.

2.2 Events:

Definition 2.2:

An event A is a subset of the sample space S. That is $A\subseteq S$.

- We say that an event A occurs if the outcome (the result) of the experiment is an element of A.
- $\phi\subseteq S$ is an event (ϕ is called the impossible event)
- $S\subseteq S$ is an event (S is called the sure event)

Example:

Experiment: Selecting a ball from a box containing 6 balls numbered 1,2,3,4,5 and 6. (or tossing a die)

- This experiment has 6 possible outcomes
  The sample space is $S=\{1,2,3,4,5,6\}$.
- Consider the following events:

E_1=getting an even number $=\{2,4,6\}\subseteq S$

E_2 =getting a number less than $4=\{1,2,3\}\subseteq S$

E_3 =getting 1 or $3=\{1,3\}\subseteq S$

E_4 =getting an odd number$=\{1,3,5\}\subseteq S$

E_5 =getting a negative number$=\{\ \}=\phi\subseteq S$

E_6 =getting a number less than 10 $= \{1,2,3,4,5,6\}=S\subseteq S$

Notation:

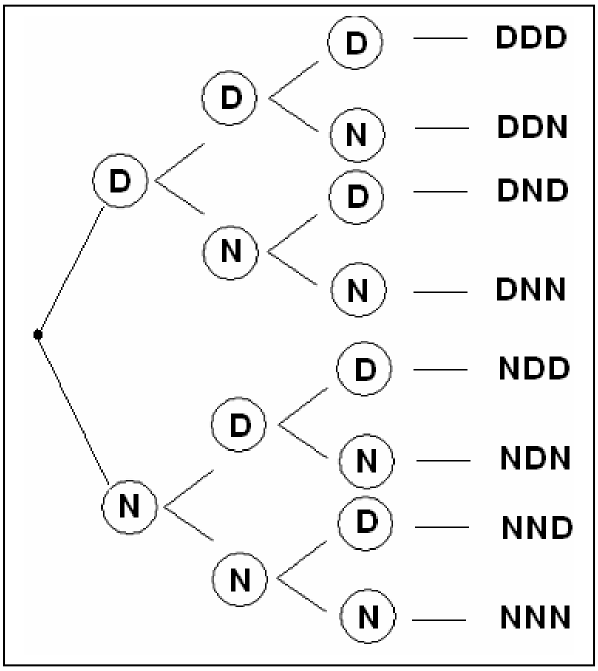
$n(S)$= no. of outcomes (elements) in S.

$n(E)$= no. of outcomes (elements) in the event E.

Example:

Experiment: Selecting 3 items from manufacturing process; each item is inspected and classified as defective (D) or non-defective (N).

- This experiment has 8 possible outcomes
  S={DDD,DDN,DND,DNN,NDD,NDN,NND,NNN}



- Consider the following events:

A={at least 2 defectives}= {DDD,DDN,DND,NDD}$\subseteq S$

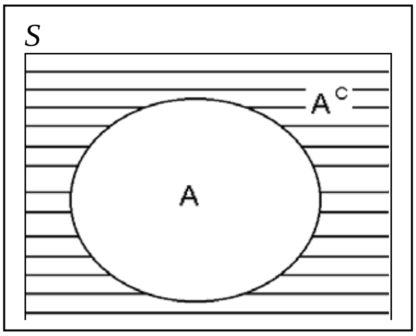
B={at most one defective}={DNN,NDN,NND,NNN}$\subseteq S$

C={3 defectives}={DDD}$\subseteq S$

Some Operations on Events:

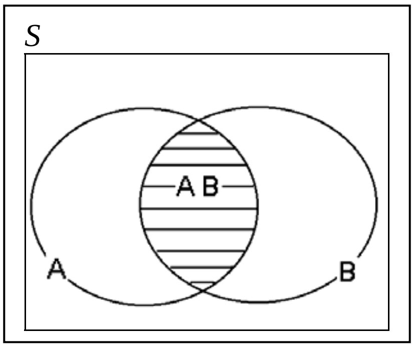
Let A and B be two events defined on the sample space S.

Definition 2.3: Complement of The Event *A*:



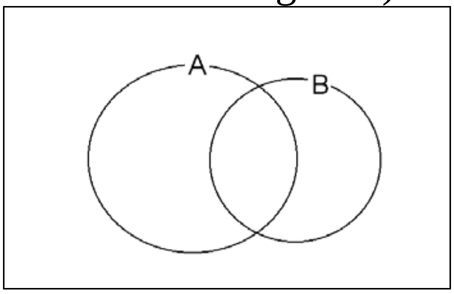
- $\mathbf{A^c}$ **or** $\mathbf{A'}$ **or** $\overline{A}$
- $A^c = \{x \in S:\ x \notin A\}$
- A^c consists of all points of S that are not in A.
- A^c occurs if A does not.

Definition 2.4: Intersection:

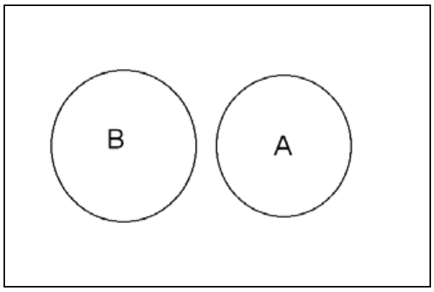


- $A \cap B = AB = \{x \in S:\ x \in A \text{ and } x \in B\}$
- $A \cap B$ Consists of all points in both A **and** B.
- $A \cap B$ Occurs if both A **and** B occur together.

Definition 2.5: Mutually Exclusive (Disjoint) Events:
Two events *A* and *B* are mutually exclusive (or disjoint) if and only if $A \cap B = \phi$; that is, *A* and *B* have no common elements (they do not occur together).



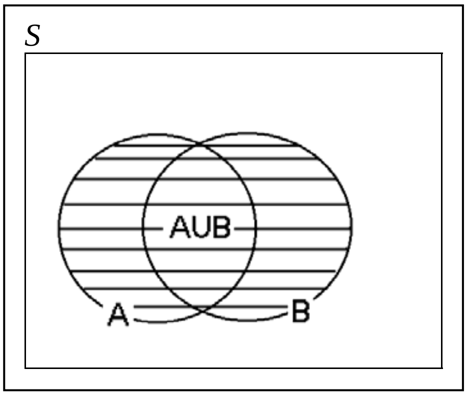
$A \cap B \neq \phi$
A and *B* are not mutually exclusive



$A \cap B = \phi$
A and *B* are mutually exclusive (disjoint)

Definition 2.6: Union:

- $A \cup B = \{x \in S: x \in A \text{ or } x \in B\}$
- $A \cup B$ Consists of all outcomes in *A* **or** in *B* **or** in both *A* and *B*.
- $A \cup B$ Occurs if *A* occurs, **or** *B* occurs, **or** both *A* and *B* occur. That is $A \cup B$ Occurs if at least one of *A* and *B* occurs.



<u>2.3 Counting Sample Points:</u>

- There are many counting techniques which can be used to count the number points in the sample space (or in some events) without listing each element.
- In many cases, we can compute the probability of an event by using the counting techniques.

Combinations:
In many problems, we are interested in the number of ways of selecting *r* objects from *n* objects without regard to order. These selections are called combinations.

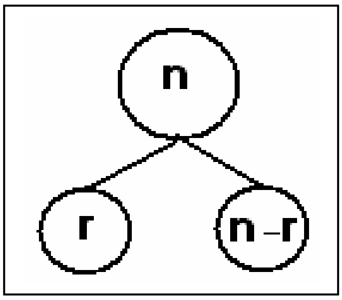
- Notation:
  n factorial is denoted by $n!$ and is defined by:
  $$n! = n \times (n-1) \times (n-2) \times \cdots \times (2) \times (1) \qquad for \quad n = 1, 2, \cdots$$
  $$0! = 1$$

Example: $5! = 5\times 4\times 3\times 2\times 1 = 120$

Theorem 2.8:

The number of combinations of n distinct objects taken r at a time is denoted by $\binom{n}{r}$ and is given by:

$$\binom{n}{r} = \frac{n!}{r!\ (n-r)!}; \qquad r = 0, 1, 2, \ldots, n$$



Notes:

- $\binom{n}{r}$ is read as " n " choose " r ".
- $\binom{n}{n} = 1$, $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{r} = \binom{n}{n-r}$
- $\binom{n}{r}$ = The number of different ways of selecting r objects from n distinct objects.
- $\binom{n}{r}$ = The number of different ways of dividing n distinct objects into two subsets; one subset contains r objects and the other contains the rest (n–r) bjects.

Example:

If we have 10 equal–priority operations and only 4 operating rooms are available, in how many ways can we choose the 4 patients to be operated on first?

Solution:

$n = 10 \qquad r = 4$

The number of different ways for selecting 4 patients from 10 patients is

$$\binom{10}{4} = \frac{10!}{4!\,(10-4)!} = \frac{10!}{4!\times 6!} = \frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{(4\times 3\times 2\times 1)\ \times\ (6\times 5\times 4\times 3\times 2\times 1)}$$

$$= 210 \qquad (\textit{different ways})$$

<u>2.4. Probability of an Event:</u>

- To every point (outcome) in the sample space of an experiment S, we assign a weight (or probability), ranging from 0 to 1, such that the sum of all weights (probabilities) equals 1.

- The weight (or probability) of an outcome measures its likelihood (chance) of occurrence.
- To find the probability of an event *A*, we sum all probabilities of the sample points in *A*. This sum is called the probability of the event *A* and is denoted by P(*A*).

Definition 2.8:

The probability of an event *A* is the sum of the weights (probabilities) of all sample points in *A*. Therefore,

1. $0 \le P(A) \le 1$
2. $P(S) = 1$
3. $P(\phi) = 0$

Example 2.22:

A balanced coin is tossed twice. What is the probability that at least one head occurs?

Solution:

S = {HH, HT, TH, TT}

A = {at least one head occurs}= {HH, HT, TH}

Since the coin is balanced, the outcomes are equally likely; i.e., all outcomes have the same weight or probability.

Outcome	Weight (Probability)
HH	P(HH) = w
HT	P(HT) = w
TH	P(TH) = w
TT	P(TT) = w
sum	4w=1

4w =1 ⇔ w =1/4 = 0.25

P(HH)=P(HT)=P(TH)=P(TT)=0.25

The probability that at least one head occurs is:

P(*A*) = P({at least one head occurs})=P({HH, HT, TH})

= P(HH) + P(HT) + P(TH)

= 0.25+0.25+0.25

= 0.75

Theorem 2.9:

If an experiment has *n*(*S*)=*N* equally likely different outcomes, then the probability of the event *A* is:

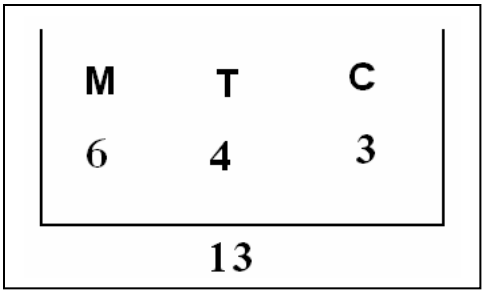
$$P(A) = \frac{n(A)}{n(S)} = \frac{n(A)}{N} = \frac{no.\ of\ outcomes\ \ in\ \ A}{no.\ of\ outcomes\ \ in\ \ S}$$

Example 2.25:

A mixture of candies consists of 6 mints, 4 toffees, and 3 chocolates. If a person makes a random selection of one of these candies, find the probability of getting:

(a) a mint

(b) a toffee or chocolate.



Solution:

Define the following events:

M = {getting a mint}

T = {getting a toffee}

C = {getting a chocolate}

Experiment: selecting a candy at random from 13 candies

$n(S)$ = no. of outcomes of the experiment of selecting a candy.

= no. of different ways of selecting a candy from 13 candies.

$= \binom{13}{1} = 13$

The outcomes of the experiment are equally likely because the selection is made at random.

(a) M = {getting a mint}

$n(M)$ = no. of different ways of selecting a mint candy from 6 mint candies

$= \binom{6}{1} = 6$

P(M)= P({getting a mint})=$\frac{n(M)}{n(S)} = \frac{6}{13}$

(b) $T \cup C$ = {getting a toffee or chocolate}

$n(T \cup C)$ = no. of different ways of selecting a toffee **or** a chocolate candy

= no. of different ways of selecting a toffee candy **+** no. of different ways of selecting a chocolate candy

$= \binom{4}{1} + \binom{3}{1} = 4 + 3 = 7$

= no. of different ways of selecting a candy from 7 candies

$$= \binom{7}{1} = 7$$

P($T \cup C$)= P({getting a toffee or chocolate})= $\frac{n(T \cup C)}{n(S)} = \frac{7}{13}$

Example 2.26:

In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

Solution:

Experiment: selecting 5 cards from 52 cards.

$n(S)$ = no. of outcomes of the experiment of selecting 5 cards from 52 cards.

$$= \binom{52}{5} = \frac{52!}{5! \times 47!} = 2598960$$

The outcomes of the experiment are equally likely because the selection is made at random.

Define the event A = {holding 2 aces **and** 3 jacks}

$n(A)$ = no. of ways of selecting 2 aces **and** 3 jacks

= (no. of ways of selecting 2 aces) × (no. of ways of selecting 3 jacks)

= (no. of ways of selecting 2 aces from 4 aces) × (no. of ways of selecting 3 jacks from 4 jacks)

$$= \binom{4}{2} \times \binom{4}{3}$$

$$= \frac{4!}{2! \times 2!} \times \frac{4!}{3! \times 1!} = 6 \times 4 = 24$$

P(A)= P({holding 2 aces **and** 3 jacks })

$$= \frac{n(A)}{n(S)} = \frac{24}{2598960} = 0.000009$$

2.5 Additive Rules:

Theorem 2.10:

If A and B are any two events, then:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Corollary 1:

If A and B are mutually exclusive (disjoint) events, then:

$$P(A \cup B) = P(A) + P(B)$$

Corollary 2:

If $A_1, A_2, \ldots, A_n$ are n mutually exclusive (disjoint) events, then:

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Note: Two event Problems:

* In Venn diagrams, consider the probability of an event A as the area of the region corresponding to the event A.

* Total area= $P(S)=1$

* Examples:

$P(A) = P(A \cap B) + P(A \cap B^C)$

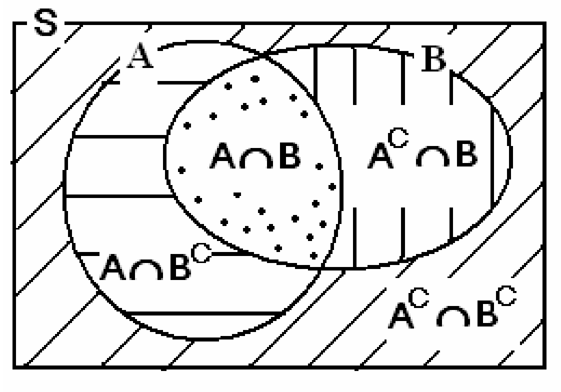
$P(A \cup B) = P(A) + P(A^C \cap B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B^C) = P(A) - P(A \cap B)$

$P(A^C \cap B^C) = 1 - P(A \cup B)$

etc.,



Total area= $P(S)=1$

Example 2.27:

The probability that Paula passes Mathematics is 2/3, and the probability that she passes English is 4/9. If the probability that she passes both courses is 1/4, what is the probability that she will:

(a) pass at least one course?

(b) pass Mathematics and fail English?

(c) fail both courses?

Solution:

Define the events: M={Paula passes Mathematics}

E={Paula passes English}

We know that $P(M)=2/3$, $P(E)=4/9$, and $P(M \cap E)=1/4$.

(a) Probability of passing at least one course is:

$$P(M \cup E) = P(M) + P(E) - P(M \cap E)$$

$$= \frac{2}{3} + \frac{4}{9} - \frac{1}{4} = \frac{31}{36}$$

(b) Probability of passing Mathematics and failing English is:

$$P(M \cap E^C) = P(M) - P(M \cap E)$$

$$= \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

(c) Probability of failing both courses is:

$$P(M^C \cap E^C) = 1 - P(M \cup E)$$

$$= 1 - \frac{31}{36} = \frac{5}{36}$$

Theorem 2.12:

If A and A^C are complementary events, then:

$$P(A) + P(A^C) = 1 \iff P(A^C) = 1 - P(A)$$

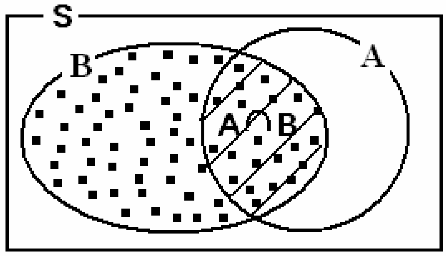
2.6 Conditional Probability:

The probability of occurring an event A when it is known that some event B has occurred is called the conditional probability of A given B and is denoted $P(A|B)$.

Definition 2.9:

The conditional probability of the event A given the event B is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad ; \; P(B) > 0$$



P(S)=Total area=1

Notes:

1. $P(A|B) = \frac{P(A \cap B)}{P(B)} =$

$$= \frac{n(A \cap B)/n(S)}{n(B)/n(S)} = \frac{n(A \cap B)}{n(B)}; \; \textit{for equally likely outcomes case}$$

2. $P(B|A) = \frac{P(A \cap B)}{P(A)}$

3. $P(A \cap B) = P(A)\,P(B|A)$
$= P(B)\,P(A|B)$ (Multiplicative Rule=Theorem 2.13)

Example:

339 physicians are classified as given in the table below. A physician is to be selected at random.

(1) Find the probability that:

(a) the selected physician is aged 40 – 49

(b) the selected physician smokes occasionally

(c) the selected physician is aged 40 – 49 and smokes occasionally

(2) Find the probability that the selected physician is aged 40 – 49 given that the physician smokes occasionally.

		Smoking Habit			
		Daily (B_1)	Occasionally (B_2)	Not at all (B_3)	Total
Age	20 - 29 (A_1)	31	9	7	47
	30 - 39 (A_2)	110	30	49	189
	40 - 49 (A_3)	29	21	29	79
	50+ (A_4)	6	0	18	24
	Total	176	60	103	339

Solution:

.$n(S) = 339$ equally likely outcomes.

Define the following events:

A_3 = the selected physician is aged 40 – 49

B_2 = the selected physician smokes occasionally

$A_3 \cap B_2$ = the selected physician is aged 40 – 49 and smokes occasionally

(1) (a) A_3 = the selected physician is aged 40 – 49

$$P(A_3) = \frac{n(A_3)}{n(S)} = \frac{79}{339} = 0.2330$$

(b) B_2 = the selected physician smokes occasionally

$$P(B_2) = \frac{n(B_2)}{n(S)} = \frac{60}{339} = 0.1770$$

(c) $A_3 \cap B_2$ = the selected physician is aged 40 – 49 and smokes occasionally.

$$P(A_3 \cap B_2) = \frac{n(A_3 \cap B_2)}{n(S)} = \frac{21}{339} = 0.06195$$

(2) $A_3|B_2$ = the selected physician is aged 40 – 49 given that the physician smokes occasionally

(i) $P(A_3 \mid B_2) = \dfrac{P(A_3 \cap B_2)}{P(B_2)} = \dfrac{0.06195}{0.1770} = 0.35$

(ii) $P(A_3 \mid B_2) = \dfrac{n(A_3 \cap B_2)}{n(B_2)} = \dfrac{21}{60} = 0.35$

(iii) We can use the restricted table directly: $P(A_3 \mid B_2) = \dfrac{21}{60} = 0.35$

Notice that $P(A_3|B_2)=0.35 > P(A_3)=0.233$.
The conditional probability does not equal unconditional probability; i.e., $P(A_3|B_2) \neq P(A_3)$! What does this mean?

Note:

- $P(A|B)=P(A)$ means that knowing B has no effect on the probability of occurrence of A. In this case A is independent of B.
- $P(A|B)>P(A)$ means that knowing B increases the probability of occurrence of A.
- $P(A|B)<P(A)$ means that knowing B decreases the probability of occurrence of A.

Independent Events:

Definition 2.10:

Two events A and B are independent if and only if $P(A|B)=P(A)$ and $P(B|A)=P(B)$. Otherwise A and B are dependent.

Example:

In the previous example, we found that $P(A_3|B_2) \neq P(A_3)$. Therefore, the events A_3 and B_2 are dependent, i.e., they are not independent. Also, we can verify that $P(B_2| A_3) \neq P(B_2)$.

2.7 Multiplicative Rule:

Theorem 2.13:

If $P(A) \neq 0$ and $P(B) \neq 0$, then:

$$P(A\cap B) = P(A)\, P(B|A)$$
$$= P(B)\, P(A|B)$$

Example 2.32:

Suppose we have a fuse box containing 20 fuses of which 5 are defective (D) and 15 are non-defective (N). If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Solution:

Define the following events:

A = {the first fuse is defective}

B = {the second fuse is defective}

$A\cap B$={the first fuse is defective and the second fuse is defective} = {both fuses are defective}

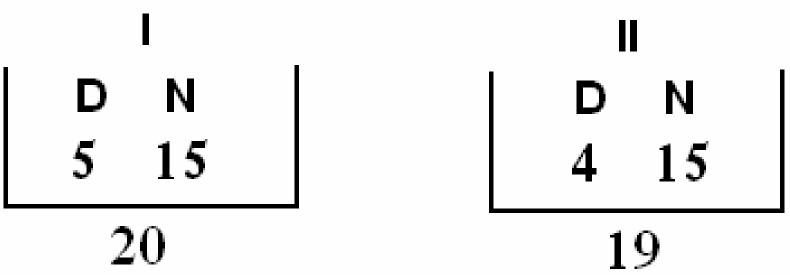
We need to calculate P($A \cap B$).

$P(A) = \frac{5}{20}$

$P(B|A) = \frac{4}{19}$

$P(A \cap B) = P(A)\,P(B|A)$

$= \frac{5}{20} \times \frac{4}{19} = 0.052632$



First Selection

Second Selection: given that the first is defective (D)

Theorem 2.14:

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)\,P(B)$$

*(Multiplicative Rule for independent events)

Note:

Two events A and B are independent if one of the following conditions is satisfied:

(i) $P(A|B)=P(A)$

$\Leftrightarrow$ (ii) $P(B|A)=P(B)$

$\Leftrightarrow$ (iii) $P(A \cap B) = P(A)\,P(B)$

Theorem 2.15: (k=3)

- If A_1, A_2, A_3 are 3 events, then:

  $$P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 \cap A_2)$$

- If A_1, A_2, A_3 are 3 independent events, then:

  $$P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3)$$

Example 2.36:

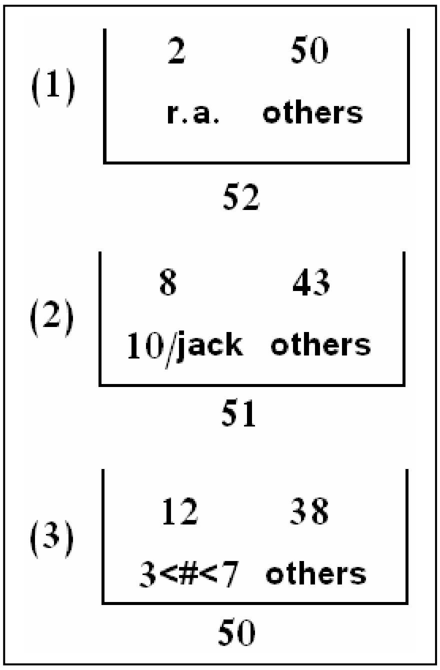
Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Fined $P(A_1 \cap A_2 \cap A_3)$, where the events A_1, A_2, and A_3 are defined as follows:

A_1 = {the 1-st card is a red ace}

A_2 = {the 2-nd card is a 10 or a jack}

A_3 = {the 3-rd card is a number greater than 3 but less than 7}

Solution:



$P(A_1) = 2/52$

$P(A_2 | A_1) = 8/51$

$P(A_3 | A_1 \cap A_2) = 12/50$

$P(A_1 \cap A_2 \cap A_3)$

$= P(A_1)\, P(A_2 | A_1)\, P(A_3 | A_1 \cap A_2)$

$= \frac{2}{52} \times \frac{8}{51} \times \frac{12}{50}$

$= \frac{192}{132600}$

$= 0.0014479$

2.8 Bayes' Rule:

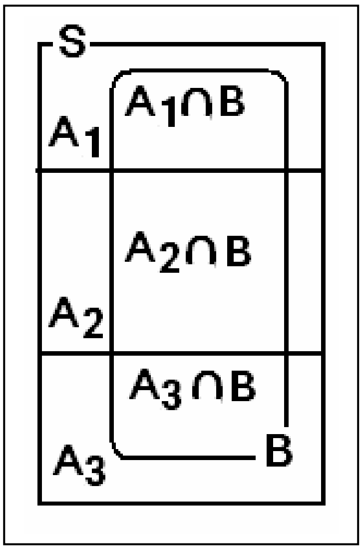
Definition:

The events $A_1, A_2, \ldots,$ and A_n constitute a partition of the sample space S if:

- $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = S$
- $A_i \cap A_j = \phi, \quad \forall\, i \neq j$

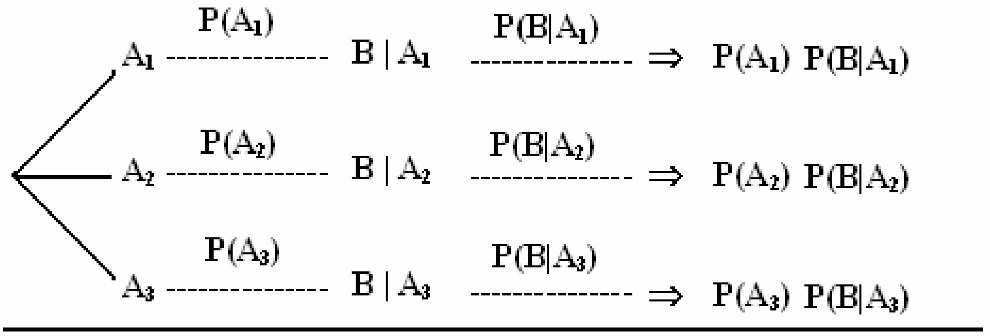
Theorem 2.16: (Total Probability)

If the events $A_1, A_2, \ldots,$ and A_n constitute a partition of the sample space S such that $P(A_k) \neq 0$ for $k = 1, 2, \ldots, n$, then for any event B:



$$P(B) = \sum_{k=1}^{n} P(A_k \cap B)$$

$$= \sum_{k=1}^{n} P(A_k) P(B | A_k)$$



$$\text{المجموع} = P(B) = \sum_{k=1}^{n} P(A_k)\, P(B \mid A_k)$$

Tree Diagram

<u>Example 2.38:</u>

Three machines A_1, A_2, and A_3 make 20%, 30%, and 50%, respectively, of the products. It is known that 1%, 4%, and 7% of the products made by each machine, respectively, are defective. If a finished product is randomly selected, what is the probability that it is defective?

Solution:

Define the following events:

B = {the selected product is defective}

A_1 = {the selected product is made by machine A_1}

A_2 = {the selected product is made by machine A_2}

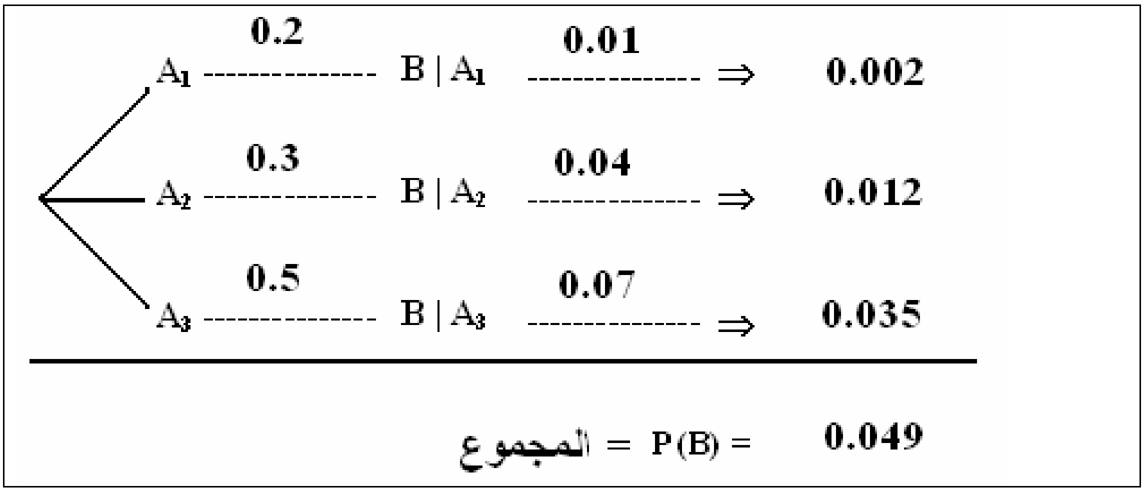
A_3 = {the selected product is made by machine A_3}

$P(A_1) = \frac{20}{100} = 0.2$; $P(B|A_1) = \frac{1}{100} = 0.01$

$P(A_2) = \frac{30}{100} = 0.3$; $P(B|A_2) = \frac{4}{100} = 0.04$

$P(A_3) = \frac{50}{100} = 0.5$; $P(B|A_3) = \frac{7}{100} = 0.07$

$$\begin{aligned}
P(B) &= \sum_{k=1}^{3} P(A_k)P(B \mid A_k) \\
&= P(A_1)\,P(B|A_1) + P(A_2)\,P(B|A_2) + P(A_3)\,P(B|A_3) \\
&= 0.2\times0.01 + 0.3\times0.04 + 0.5\times0.07 \\
&= 0.002 \quad + 0.012 \quad + 0.035 \\
&= 0.049
\end{aligned}$$



Question:

If it is known that the selected product is defective, what is the probability that it is made by machine A_1?

Answer:

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(A_1)P(B|A_1)}{P(B)} = \frac{0.2\times 0.01}{0.049} = \frac{0.002}{0.049} = 0.0408$$

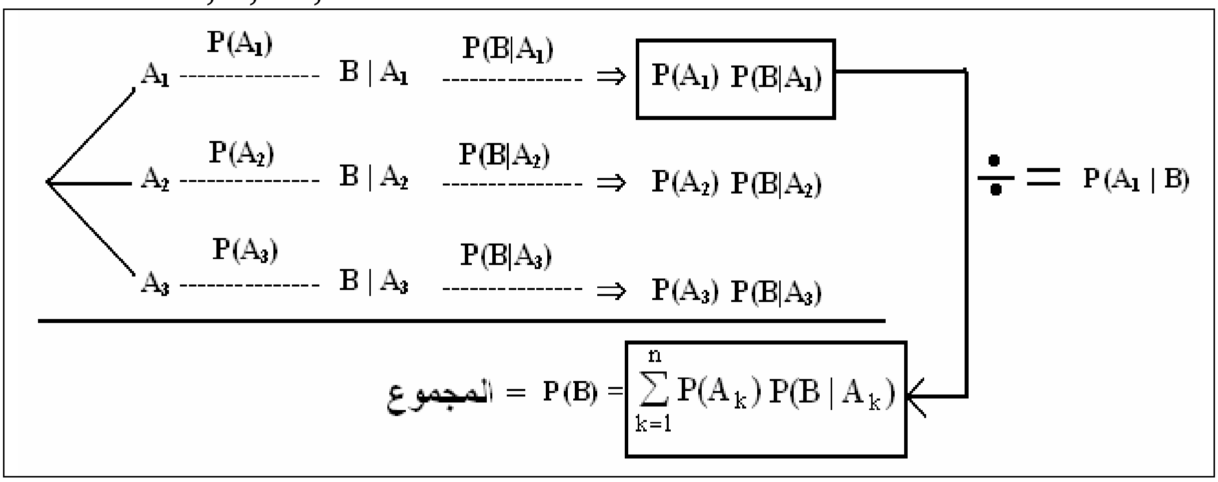
This rule is called Bayes' rule.

Theorem 2.17: (Bayes' rule)

If the events $A_1, A_2, \ldots,$ and A_n constitute a partition of the sample space S such that $P(A_k)\neq 0$ for $k=1, 2, \ldots, n$, then for any event B such that $P(B)\neq 0$:

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)} = \frac{P(A_i)P(B|A_i)}{P(B)}$$

for i = 1, 2, …, n.



Example 2.39:

In Example 2.38, if it is known that the selected product is defective, what is the probability that it is made by:

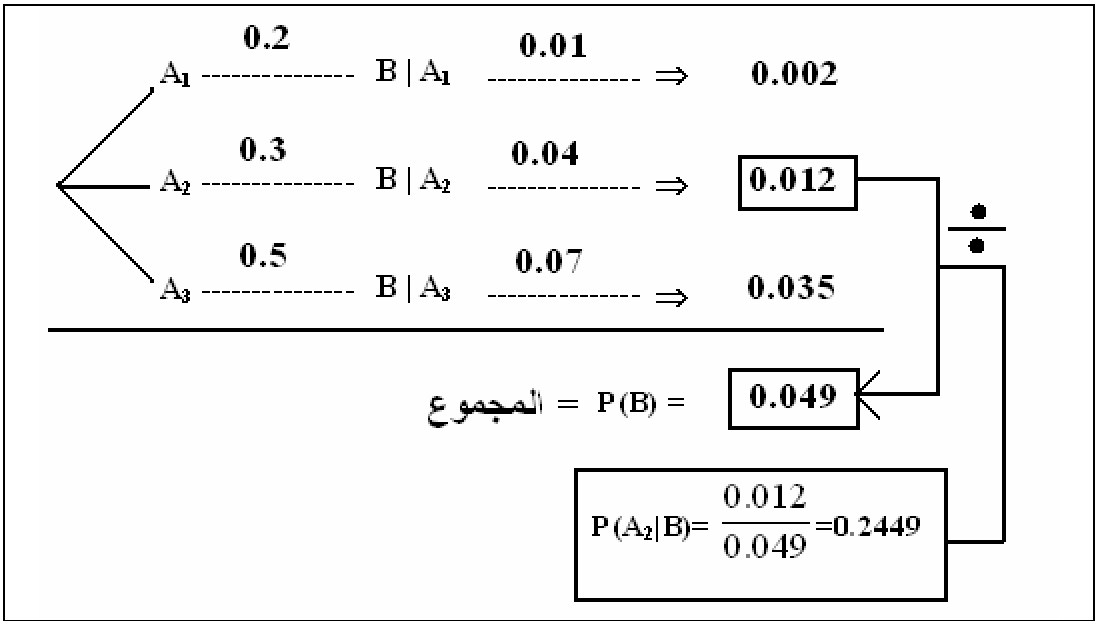
(a) machine A_2?

(b) machine A_3?

Solution:

(a) $P(A_2|B)=\dfrac{P(A_2)P(B|A_2)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)}=\dfrac{P(A_2)P(B|A_2)}{P(B)}$

$=\dfrac{0.3\times 0.04}{0.049}=\dfrac{0.012}{0.049}=0.2449$



(b) $P(A_3|B)=\dfrac{P(A_3)P(B|A_3)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)}=\dfrac{P(A_3)P(B|A_3)}{P(B)}$

$=\dfrac{0.5\times 0.07}{0.049}=\dfrac{0.035}{0.049}=0.7142$

Note:

$P(A_1|B) = 0.0408$, $P(A_2|B) = 0.2449$, $P(A_3|B) = 0.7142$

- $\sum_{k=1}^{3} P(A_k|B)=1$
- If the selected product was found defective, we should check machine A_3 first, if it is ok, we should check machine A_2, if it is ok, we should check machine A_1.