Chapter 3: Functions

3.1. Definition of Function

Functions arise whenever one quantity depends on another. Consider the following situation:

- The area of a circle depends on the radius of the circle. The rule that connects *r* and *A* is given by the equation $A = \pi r^2$. With each positive number *r* there is associated one value of *A*, and we say that ***A* is a *function* of *r***.

A function *f* is a rule that assigns to each element *x* in a set *A* exactly one element, called *f(x)*, in a set *B*.

We usually consider functions for which the sets *A* and *B* are sets of real numbers. The Set *A* is called the **domain** of the function. The number *f(x)* is the value of *f* at *x* and is read **"*f of x.*"** The **range** of *f* is the set of all possible values of *f(x)* as *x* varies throughout the domain.

It's helpful to think of a function as a **machine** (see **Figure 1**). If *x* is in the domain of the function *f* then when *x* enters the machine, it's accepted as an input and the machine produces an output according to the rule of the function. Thus, we can think of the **domain** as the set of all possible inputs and the **range** as the set of all possible outputs. The preprogrammed functions in a calculator are good examples of a function as a machine. For example, the square root key on your calculator computes such a function. You press the key labeled $\sqrt{\ }$ (or) $\sqrt{x}$ and enter the input *x*. If $x \prec 0$, then *x* is not in the domain of this function; that is, is not an acceptable input, and the calculator will indicate an error. If $x \geq 0$, then the calculator will give a value for $\sqrt{x}$.

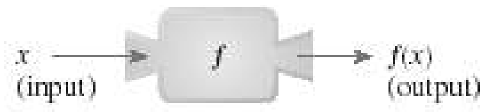


Figure 1: Machine Diagram for a function *f*

Another way to picture a function is by an **arrow diagram** as in **Figure 2**. Each arrow connects an element of *A* to an element of *B*. The arrow indicates that *f(x)* is associated with *x*, *f(a)* is associated with *a*, and so on.

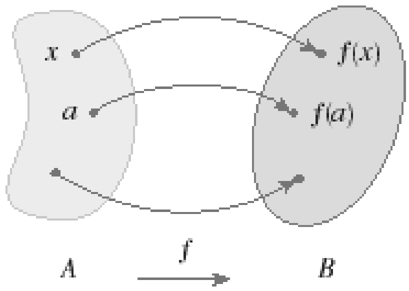


Figure 2: Arrow Diagram for *f*

3.2. Graph of Function

The most common method for visualizing a function is its graph. If is a function with domain, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

In other words, the graph of *f* consists of all points *(x, y)* in the coordinate plane such that *y = f(x)* and *x* is in the domain of *f*.

We can read the value of *f(x)* from the graph as being the height of the graph above the point *x* (see **Figure 3**). The graph of *f* also allows us to picture the domain of *f* on the *x-axis* and its range on the *y-axis* as in **Figure 4**.

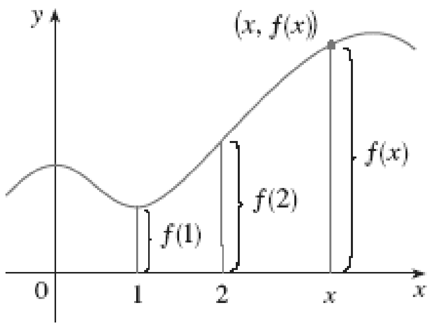


Figure 3

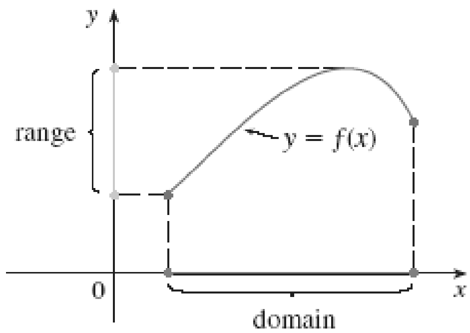


Figure4

EXAMPLE 1: The graph of a function *f* is shown in **Figure 5**.

(a) Find the values of *f(1)* and *f(5)*.

(b) What are the domain and range of *f* ?

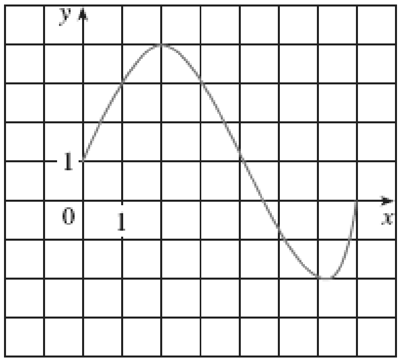


Figure 5

SOLUTION

(a) We see from **Figure 5** that the point (1, 3) lies on the graph of f, so the value of f at 1 is f(1) = 3. (In other words, the point on the graph that lies above x = 1 is 3 units above the x-axis.)

When x = 5, the graph lies about 0.7 unit below the x-axis, so we estimate that

$f(5) \approx -0.7$

(b) We see that f(x) is defined when $0 \leq x \leq 7$, so the domain of f is the closed interval [0, 7]. Notice that f takes on all values from -2 to 4, so the range of f is

$$\{y \mid -2 \leq y \leq 4\} = [-2, 4]$$

EXAMPLE 2: Sketch the graph and find the domain and range of each function.

$(a)\ f(x) = 2x - 1$ $\qquad$ $(b)\ g(x) = x^2$

SOLUTION:

(a) The equation of the graph is y = 2x - 1, and we recognize this as being the equation of a line with slope 2 and y-intercept -1.

This enables us to sketch the graph of *f* in **Figure 6**. The expression 2x -1 is defined for all real numbers, so the domain of *f* is the set of all real numbers, which we denote by $\Re$. The graph shows that the range is also $\Re$.

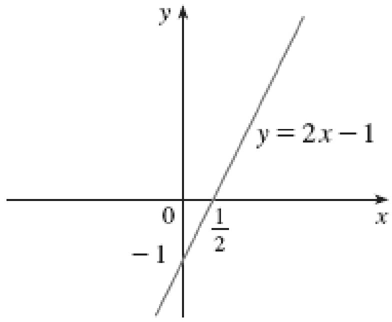


Figure 6

(b) Since $g(2) = 2^2 = 4$ and $g(-1) = -1^2 = 1$, we could plot the points (2, 4) and (-1, 1), together with a few other points on the graph, and join them to produce the graph (**Figure 7**). The equation of the graph is $y = x^2$. The domain of g is $\Re$. The range of g consists of all values of g(x), that is, all numbers of the form x^2. But $x^2 \geq 0$ for all numbers x and any positive number y is a square. So the range of g is $\{y|\ y \geq 0\} = [0, \infty)$. This can also be seen from **Figure 7**.

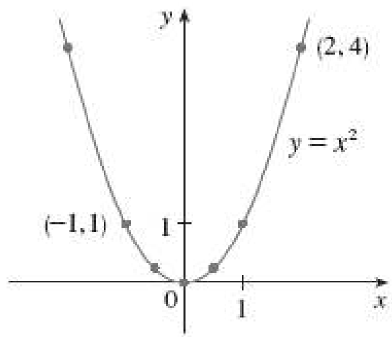


Figure 7

EXAMPLE 3: Find the domain of each function.

$(a)\ f(x) = \sqrt{x+2}$ $\qquad$ $(b)\ g(x) = \dfrac{1}{x^2 - x}$

SOLUTION

(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x+2\geq 0$. This is equivalent to $x\geq -2$, so the domain is the interval $[-2,\infty)$.

(b) Since

$$g(x)=\frac{1}{x^2-x}=\frac{1}{x(x-1)}$$

and division by 0 is not allowed, we see that g(x) is not defined when x = 0 or x = 1. Thus, the domain of g is

$$\{x \mid x\neq 0,\ x\neq 1\}$$

which could also be written in interval notation as

$$(-\infty,0)\ \cup\ (0,1)\ \cup\ (1,\infty)$$

3.3. *Piecewise Defined Functions*

The functions in the following two examples are defined by different formulas in different parts of their domains.

EXAMPLE 4: A function *f* is defined by

$$f(x)=\begin{cases}1-x & \text{if } x\leq 1\\ x^2 & \text{if } x>1\end{cases}$$

Evaluate $f(0)$, $f(1)$, and $f(2)$ and sketch the graph.

SOLUTION Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x. If it happens that $x\leq 1$, then the value of $f(x)$ is $1-x$. On the other hand, if $x>1$, then the value of $f(x)$ is x^2.

Since $0\leq 1$, we have $f(0)=1-0=1$.

Since $1\leq 1$, we have $f(1)=1-1=0$.

Since $2>1$, we have $f(2)=2^2=4$.

How do we draw the graph of f? We observe that if $x\leq 1$, then $f(x)=1-x$, so the part of the graph of f that lies to the left of the vertical line $x=1$ must coincide with the line $y=1-x$, which has slope -1 and y-intercept 1. If $x>1$, then $f(x)=x^2$, so the part of the graph of f that lies to the right of the line $x=1$ must coincide with the graph of $y=x^2$.

This enables us to sketch the graph in **Figure 8**.

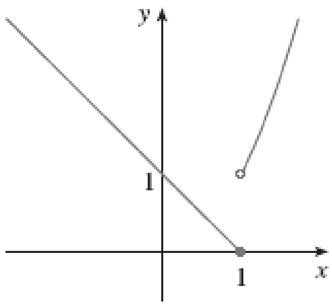


Figure 8

The solid dot indicates that the point $(1, 0)$ is included on the graph; the open dot indicates that the point $(1, 1)$ is excluded from the graph.

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geqslant 0 \qquad \text{for every number } a$$

For example,

$$|3| = 3 \qquad |-3| = 3 \qquad |0| = 0 \qquad |\sqrt{2} - 1| = \sqrt{2} - 1 \qquad |3 - \pi| = \pi - 3$$

In general, we have

$$\begin{aligned} |a| &= a && \text{if } a \geqslant 0 \\ |a| &= -a && \text{if } a < 0 \end{aligned}$$

(Remember that if a is negative, then $-a$ is positive.)

EXAMPLE 5: Sketch the graph of the absolute value function $f(x) = |x|$.

SOLUTION From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 4 we see that the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis.

(See **Figure 9**)

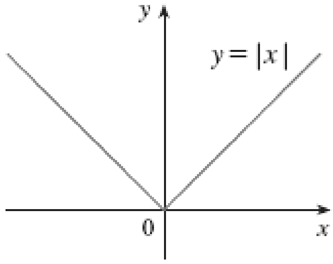


Figure 9

3.4. Symmetry

Even Function:

If a function satisfies *f(-x) = f(x)* for every number in its domain, then *f* is called an **even function**. For instance, the function $f(x) = x^2$ is even because:

$$f(-x) = (-x)^2 = x^2 = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the y-axis (see **Figure 10**). This means that if we have plotted the graph of f for $x \geq 0$, we obtain the entire graph simply by reflecting about the y-axis.

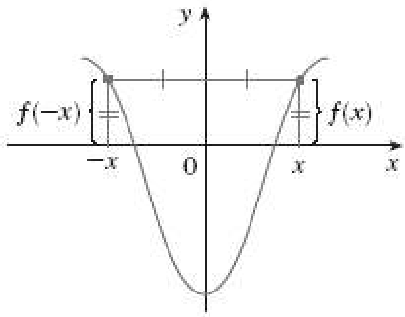


Figure 10

Odd Function:

If *f* satisfies *f(-x) = - f(x)* for every number x in its domain, then *f* is called an odd function. For example, the function $f(x) = x^3$ is odd because

$f(-x) = (-x)^3 = -x^3 = -f(x)$

The graph of an odd function is symmetric about the origin (see **Figure 11**). If we already have the graph of *f* for $x \geq 0$, we can obtain the entire graph by rotating through 180° about the origin.

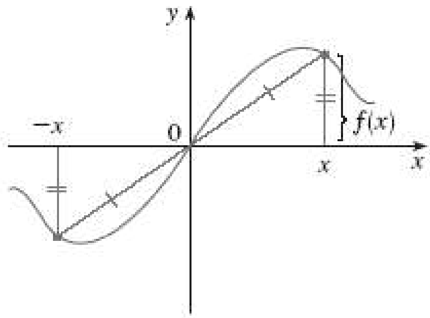


Figure 11

EXAMPLE 6:

Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

SOLUTION

(a)
$$\begin{aligned} f(-x) &= (-x)^5 + (-x) = (-1)^5 x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore, f is an odd function.

(b)
$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

(c)
$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.

3.5. Increasing and Decreasing Functions

The graph shown in **Figure 12** rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be increasing on the interval [a, b], decreasing on [b, c], and increasing again on [c, d]. Notice that if x_1 and x_2 are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$. We use this as the defining property of an increasing function.

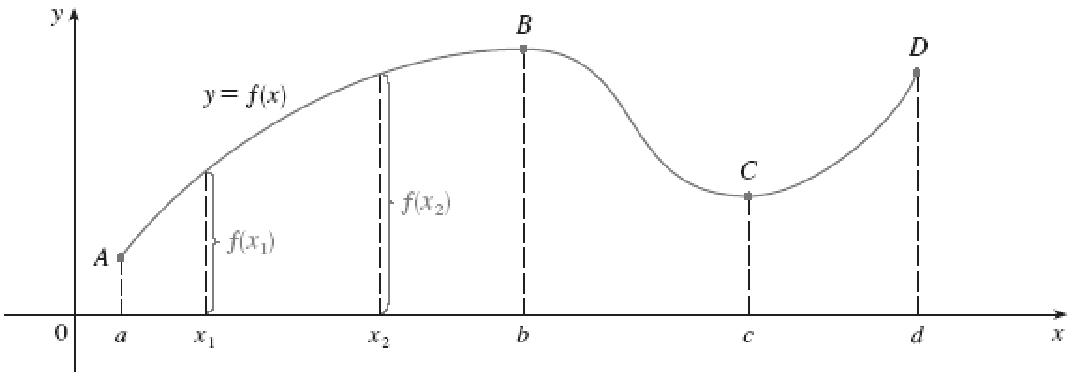


Figure 12

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

You can see from **Figure 13** that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

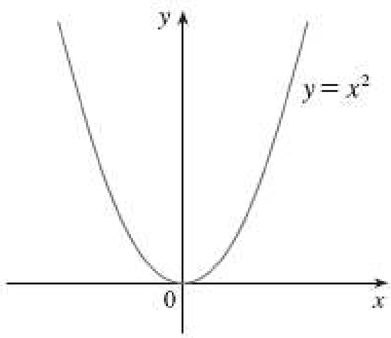


Figure 13

3.6. Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

If we define the sum $f + g$ by the equation

$$(f + g)(x) = f(x) + g(x) \tag{1}$$

then the right side of Equation 1 makes sense if both $f(x)$ and $g(x)$ are defined, that is, if x belongs to the domain of f and also to the domain of g. If the domain of f is A and the domain of g is B, then the domain of $f + g$ is the intersection of these domains, that is, $A \cap B$.

Notice that the $+$ sign on the left side of Equation 1 stands for the operation of addition of *functions*, but the $+$ sign on the right side of the equation stands for addition of the *numbers* $f(x)$ and $g(x)$.

Similarly, we can define the difference $f - g$ and the product fg, and their domains are also $A \cap B$. But in defining the quotient f/g we must remember not to divide by 0.

Algebra of Functions Let f and g be functions with domains A and B. Then the functions $f + g$, $f - g$, fg, and f/g are defined as follows:

$$(f + g)(x) = f(x) + g(x) \qquad \text{domain} = A \cap B$$

$$(f - g)(x) = f(x) - g(x) \qquad \text{domain} = A \cap B$$

$$(fg)(x) = f(x)g(x) \qquad \text{domain} = A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{domain} = \{x \in A \cap B \mid g(x) \neq 0\}$$

EXAMPLE 7:

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$, find the functions $f + g$, $f - g$, fg, and f/g.

SOLUTION The domain of $f(x) = \sqrt{x}$ is $[0, \infty)$. The domain of $g(x) = \sqrt{4 - x^2}$ consists of all numbers x such that $4 - x^2 \geq 0$, that is, $x^2 \leq 4$. Taking square roots of both sides, we get $|x| \leq 2$, or $-2 \leq x \leq 2$, so the domain of g is the interval $[-2, 2]$. The intersection of the domains of f and g is

$$[0, \infty) \cap [-2, 2] = [0, 2]$$

Thus, according to the definitions, we have

$$(f + g)(x) = \sqrt{x} + \sqrt{4 - x^2} \qquad 0 \leq x \leq 2$$

$$(f - g)(x) = \sqrt{x} - \sqrt{4 - x^2} \qquad 0 \leq x \leq 2$$

$$(fg)(x) = \sqrt{x}\sqrt{4 - x^2} = \sqrt{4x - x^3} \qquad 0 \leqslant x \leqslant 2$$

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{\sqrt{4 - x^2}} = \sqrt{\frac{x}{4 - x^2}} \qquad 0 \leqslant x < 2$$

Notice that the domain of f/g is the interval $[0, 2)$; we have to exclude $x = 2$ because $g(2) = 0$.

3.7. Composition of Functions

There is another way of combining two functions to get a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g.

In general, given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, then we can calculate the value of $f(g(x))$. The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ ("f circle g").

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

The best way to picture $f \circ g$ is by either a machine diagram (**Figure 14**) or an arrow diagram (**Figure 15**).

Figure 14: The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

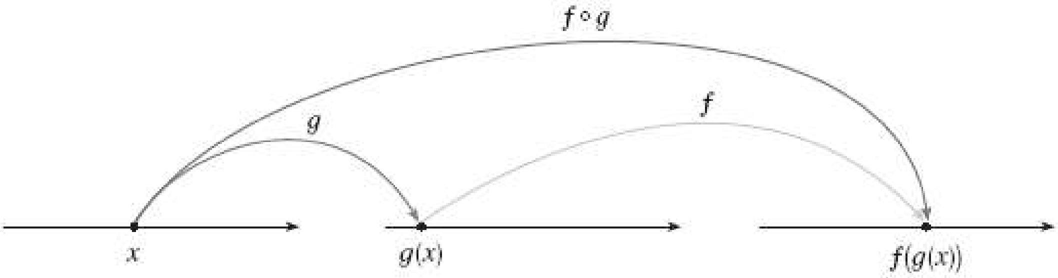


Figure 15: Arrow Diagram for $f \circ g$

EXAMPLE 8:

If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

SOLUTION We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$
$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

****** NOTE:*** You can see from Example 8 that, in general, $f \circ g \neq g \circ f$. Remember, the notation $f \circ g$ means that the function *g* is applied first and then *f* is applied second. In Example 8, $f \circ g$ is the function that first subtracts 3 and then squares; $g \circ f$ is the function that first squares and then subtracts 3.

EXAMPLE 9:

Find $f \circ g \circ h$ if $f(x) = x/(x + 1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

SOLUTION

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 3))$$
$$= f((x + 3)^{10}) = \frac{(x + 3)^{10}}{(x + 3)^{10} + 1}$$

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to decompose a complicated function into simpler ones, as in the following example.

EXAMPLE 10:

Given $F(x) = \cos^2(x+9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

SOLUTION Since $F(x) = [\cos(x+9)]^2$, the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \qquad g(x) = \cos x \qquad f(x) = x^2$$

Then

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x+9)) = f(\cos(x+9)) \\ &= [\cos(x+9)]^2 = F(x)\end{aligned}$$

EXAMPLE 11:

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find each function and its domain.

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

SOLUTION

(a) $$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{2-x}\right) = \sqrt{\sqrt{2-x}} = \sqrt[4]{2-x}$$

The domain of $f \circ g$ is $\{x \mid 2 - x \geqslant 0\} = \{x \mid x \leqslant 2\} = (-\infty, 2]$.

(b) $$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \sqrt{2-\sqrt{x}}$$

For $\sqrt{x}$ to be defined we must have $x \geqslant 0$. For $\sqrt{2-\sqrt{x}}$ to be defined we must have $2 - \sqrt{x} \geqslant 0$, that is, $\sqrt{x} \leqslant 2$, or $x \leqslant 4$. Thus, we have $0 \leqslant x \leqslant 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

(c) $$(f \circ f)(x) = f(f(x)) = f\left(\sqrt{x}\right) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

The domain of $f \circ f$ is $[0, \infty)$.

(d) $$(g \circ g)(x) = g(g(x)) = g\left(\sqrt{2-x}\right) = \sqrt{2-\sqrt{2-x}}$$

This expression is defined when $2 - x \geqslant 0$, that is, $x \leqslant 2$, and $2 - \sqrt{2-x} \geqslant 0$. This latter inequality is equivalent to $\sqrt{2-x} \leqslant 2$, or $2 - x \leqslant 4$, that is, $x \geqslant -2$. Thus, $-2 \leqslant x \leqslant 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$.

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

3.8. Classification of Functions

3.8.1. Constant Functions

The simplest of all functions are those that assign the same value to every member of the domain. These are called ***constant functions***. For example, if f is the constant function defined by *f(x) = 3,* then

$f(-1)=3,\quad f(0)=3,\quad f(\sqrt{2})=3,\quad f(9)=3$

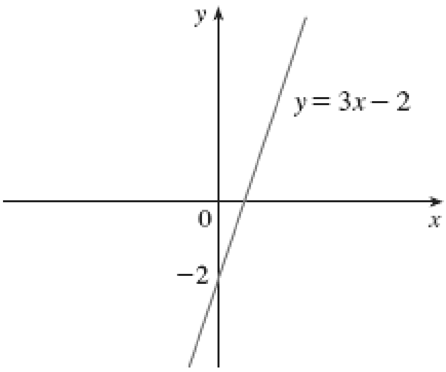
3.8.2. Linear Functions

When we say that y is a linear function of x, we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as:

$$\boxed{y=f(x)=mx+b}$$

where m is the slope of the line and b is the y-intercept.

For instance, **Figure 16** shows a graph of the linear function $f(x)=3x-2$ and a table of sample values.



x	$f(x)=3x-2$
1.0	1.0
1.1	1.3
1.2	1.6
1.3	1.9
1.4	2.2
1.5	2.5

Figure 16

3.8.3. Monomials and Polynomials

A Function of the form cx^n, where c is a constant and n is a nonnegative integer, is called a ***monomial in x***. Examples are:

$2x^3,\quad \pi x^7,\quad 4x^0(=4),\quad -6x,\quad x^{17}$

Te functions $4x^{1/2}$ and x^{-3} are not monomials because the powers of x are not nonnegative integers. A function that is expressible as the sum of finitely many monomials in x is called a ***polynomial in x.*** Examples are

$$x^3+4x+7, \quad 3-2x^3+x^{17}, \quad 9, \quad 17-\frac{2}{3}x, \quad x^5$$

The general formula for a polynomial in x is

$$\boxed{P(x)=a_n x^n + a_{n-1}x^{n-1}+.....+a_2x^2+a_1x+a_0}$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n. For example, the function

$$P(x) = 2x^6 - x^4 + \tfrac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function. A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**. Its graph is always a parabola obtained by shifting the parabola $y = ax^2$, as we will see in the next section. The parabola opens upward if $a > 0$ and downward if $a < 0$.

(See **Figure 17**.)

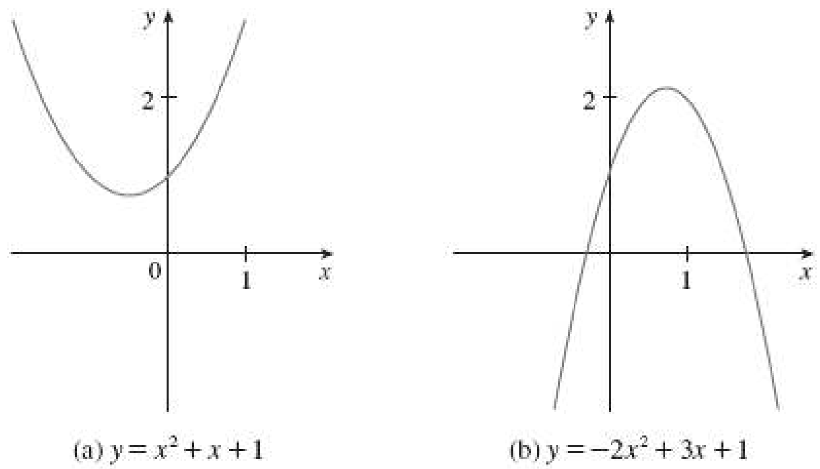


Figure 17

A polynomial of degree 3 is of the form

$$P(x)=ax^3+bx^2+cx+d$$

and is called a **cubic function**. **Figure 18** shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c). We will see later why the graphs have these shapes.

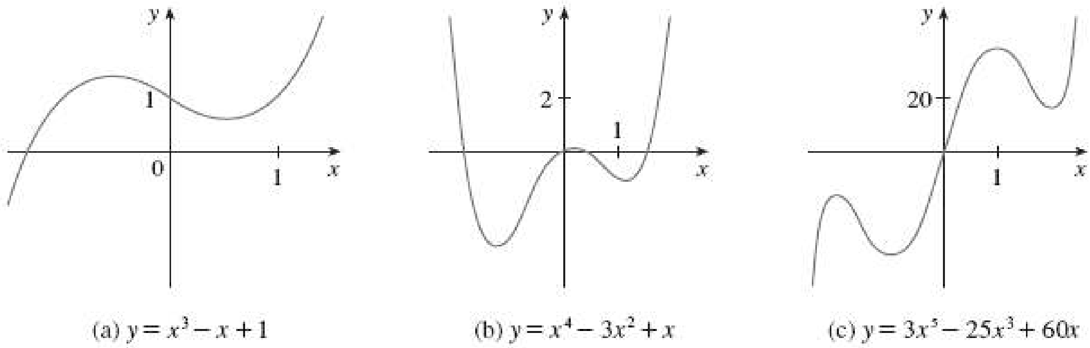


Figure 18

3.8.4. Rational Functions

A **rational function** is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

The domain consists of all values of x such that $Q(x) \neq 0$.

A simple example of a rational function is the function $f(x) = \frac{1}{x}$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in **Figure 19**.

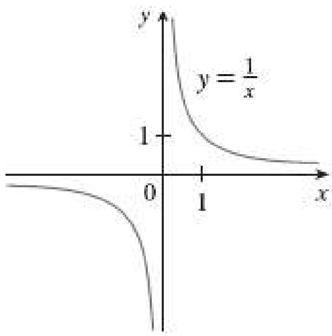


Figure 19: The reciprocal function

The function $f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$ is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in **Figure 20**.

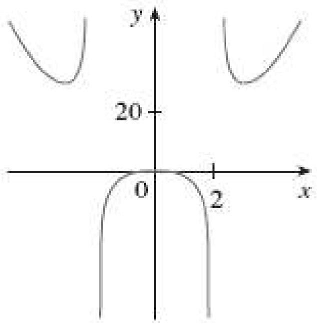
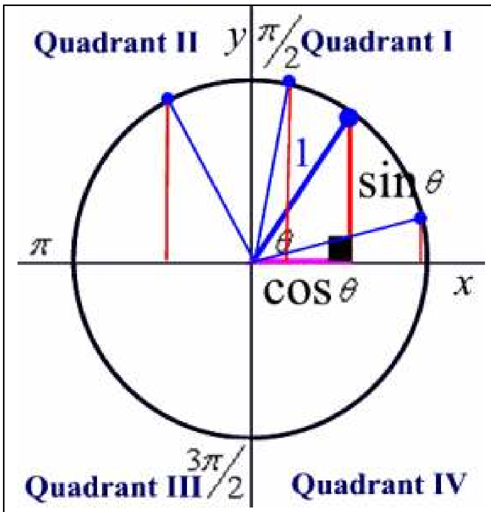
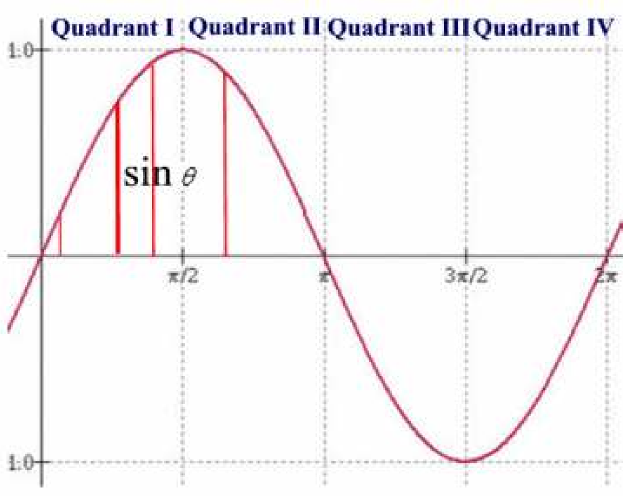


Figure 20: $f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$

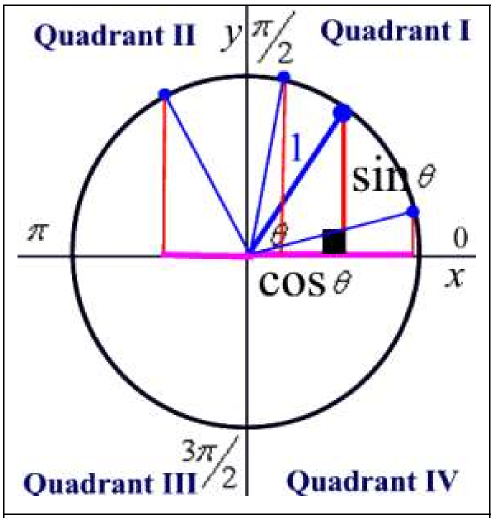
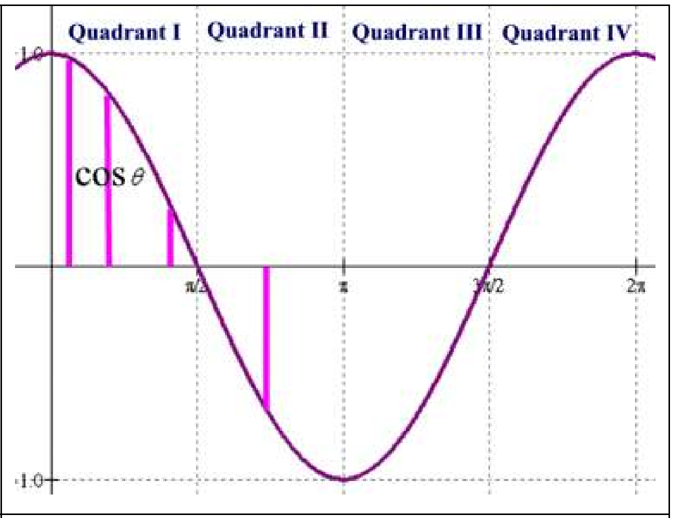
3.8.5. Trigonometric Functions

In calculus the convention is that **radian measure** is always used (except when otherwise indicated). For example, when we use the function $f(x) = \sin x$, it is understood that *sinx* means the sine of the angle whose radian measure is *x*. The **Unit Circle** shows the relationship between angles and trigonometric functions like sine and cosine. The Unit Circle is a circle with a radius of 1 centered at the origin (0, 0) in the Cartesian coordinate system. The angle that we rotate the radius uses the greek letter θ.

Generate the Sine Function:

	
Sine is represented by the vertical leg. Cosine is represented by the horizontal leg.	**When removed from the unit circle, the vertical sine segments form the graph for the sine function. Since the unit circle allows for multiple revolutions, the sine function "repeats" (a periodic function).**

Generate the Cosine Function:

	
Sine is represented by the vertical leg. **Cosine is represented by the horizontal leg.** **Notice that cosine is negative in Quadrant II.**	**When removed from the unit circle, the heights created with the horizontal cosine segments form the graph for the cosine function. Since the unit circle allows for multiple revolutions, the cosine function "repeats" (a <u>periodic function</u>).**

The following table represents the values of sin and cosine for the important angles:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
$sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0

$sin\theta$ and $cos\theta$ are the lengths of the legs of a right triangle whose hypotenuse has length 1. Thus, by the Pythagorean theorem, $sin\theta$ and $cos\theta$ satisfy the equation

$$\sin^2\theta + \cos^2\theta = 1$$

Thus, the graphs of the sine and cosine functions are as shown in **Figure 21**.

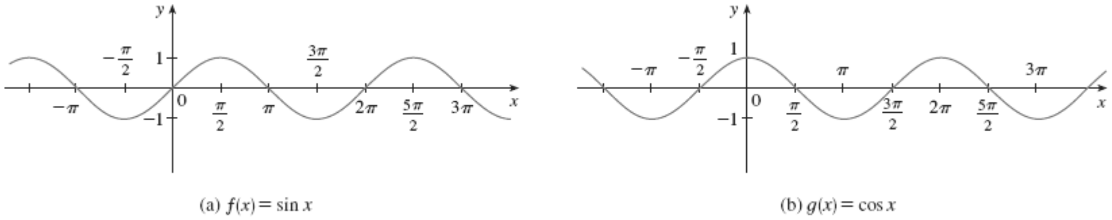


Figure 21

Notice that for both the sine and cosine functions:

- The domain is $(-\infty, \infty)$ also $\Re$

- The range is the closed interval [-1, 1]. Thus, for all values of *x*, we have

$$-1 \le \sin x \le 1 \qquad\qquad -1 \le \cos x \le 1$$

or, in terms of absolute values,

$$|\sin x| \le 1 \qquad\qquad |\cos x| \le 1$$

Also, the zeros of the sine function occur at the integer multiples of π; that is,

$$\sin x = 0 \quad \textit{when} \quad x = n\pi \quad \textit{n an integer}$$

An important property of the sine and cosine functions is that they are **periodic functions** and have period 2π. This means that, for all values of *x*,

$$\sin(x + 2\pi) = \sin x \qquad\qquad \cos(x + 2\pi) = \cos x$$

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

and its graph is shown in **Figure 22**. It is undefined whenever $\cos x = 0$, that is, when $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$. Its range is $(-\infty, \infty)$. Notice that the tangent function has period π:

$$\tan(x + \pi) = \tan x \qquad \textit{for all } x$$

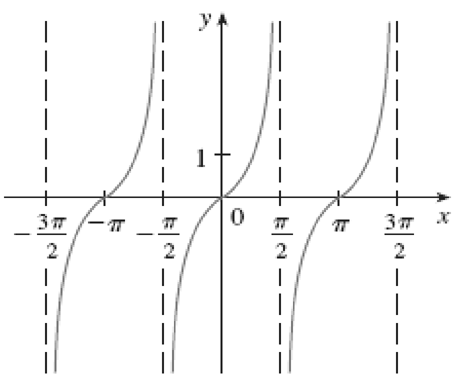


Figure 22

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions. Also:

$$\text{Cosec}\, x = \frac{1}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\text{cotan}\, x = \frac{1}{\tan x}$$

Definition of Trigonometric Functions Using Right Triangle

We can define the Trigonometric Functions Using Right Triangle as ratio of two different sides. Consider the right triangle in **Figure 23** where α denotes one of the two non-right angles. The side of the triangle opposite the right angle is called the **hypotenuse**. The **adjacent** side refers to the side that, along with the hypotenuse, forms the angle α. The third side of the triangle is called the **opposite** side.

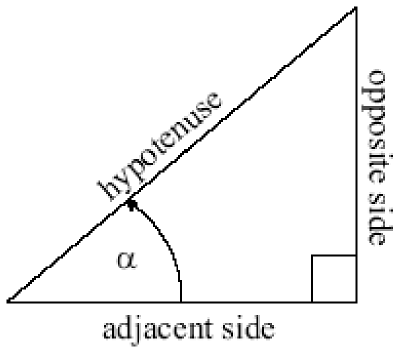


Figure 23

$$\sin\alpha = \frac{\text{opp}}{\text{hyp}} \quad \cos\alpha = \frac{\text{adj}}{\text{hyp}} \quad \tan\alpha = \frac{\text{opp}}{\text{adj}}$$

The addition formula:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

The double angle formula for *sin2A, cos2A, tan2A:*

We consider what happens if we let B equal to A. Then the first of these formulae becomes:

$\sin(A+A)=\sin A\cos A+\cos A\sin A$

so that

$$\boxed{\sin(2A)=2\sin A\cos A}$$

Similarly, if we put B equal to A in the second addition formula we have

$\cos(A+A)=\cos A\cos A+\sin A\sin A$

so that

$\boxed{\cos(2A)=\cos^2 A-\sin^2 A}$ and because $\sin^2 A+\cos^2 A=1$

we have
$$\begin{aligned}\cos(2A)&=\cos^2 A-(1-\cos^2 A)=2\cos^2 A-1\\ \cos(2A)&=(1-\sin^2 A)-\sin^2 A=1-2\sin^2 A\end{aligned}$$

so that

$$\boxed{\cos^2 A=\frac{1+\cos 2A}{2} \qquad \textit{and} \qquad \sin^2 A=\frac{1-\cos 2A}{2}}$$

3.8.6. Exponential Functions

The **exponential functions** are the functions of the form $f(x)=a^x$, where the **base** is a **positive constant**. The graphs of $y=2^x$ and $y=(0.5)^x$ are shown in **Figure 24**. In both cases the domain is $(-\infty,\infty)$ and the range is $(0,\infty)$.

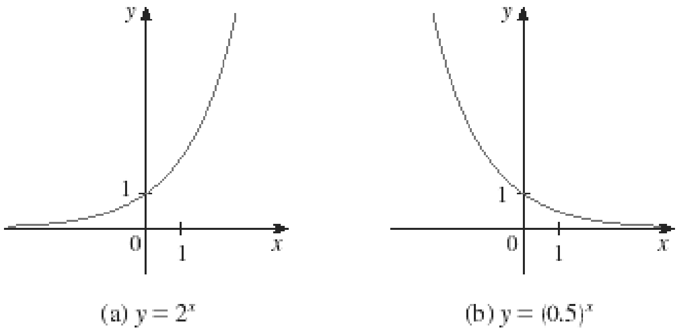


Figure 24

You can see from **Figure 25** that there are basically three kinds of exponential functions $y = a^x$. If $0 < a < 1$, the exponential function decreases; if a = 1, it is a constant; and if a > 1, it increases. These three cases are illustrated in **Figure 25**. Observe that if $a \neq 1$, then the exponential function $y = a^x$ has domain $\Re$ and range $(0, \infty)$.

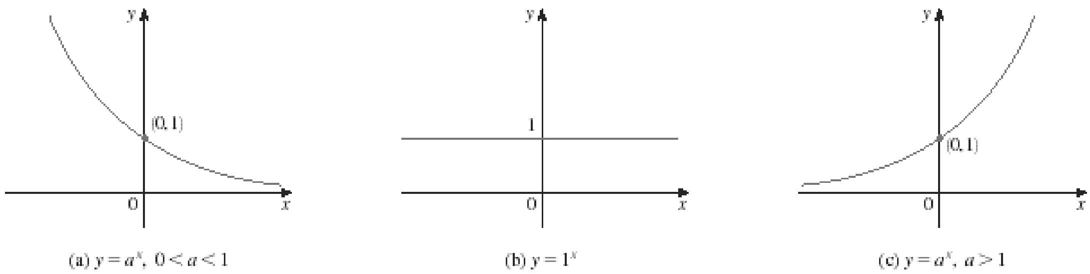


Figure 25

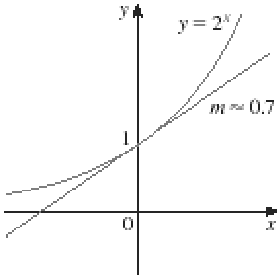
Laws of Exponents: If a and b are positive numbers and x and y are any real numbers, then

1. $a^{x+y} = a^x a^y$
2. $a^{x-y} = \dfrac{a^x}{a^y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

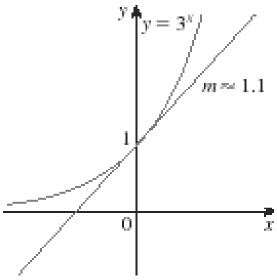
The Number e:

Figures 26 and **27** show the tangent lines to the graphs of $y = 2^x$ and $y = 3^x$ at the point (0, 1). If we measure the slopes of these tangent lines at (0, 1), we find that $m \approx 0.7$ for $y = 2^x$ and $m \approx 1.1$ for $y = 3^x$.

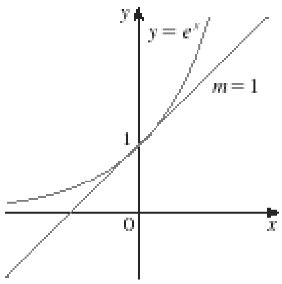
Figures 26



Figures 27

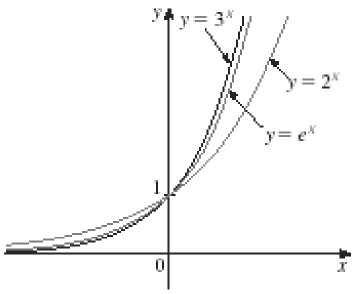


We try to determine the base a so that the slope of the tangent line to $y = a^x$ at (0, 1) is exactly 1 (see **Figure 28**). In fact, there is such a number and it is denoted by the letter e and is also known as Euler's number. (This notation was chosen by the Swiss mathematician Leonhard Euler in 1727).



Figures 28: The natural exponential function crosses the y-axis with a slope of 1.

In view of Figures 26 and 27, it comes as no surprise that the number e lies between 2 and 3 and the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. (See **Figure 29**.)



Figures 29

The value of e, correct to five decimal places, is

e **= 2.718281828**

The **exponential function** with the **base e** is called the **natural exponential function** and is written in the form e^x.

3.8.7. Logarithmic Functions

The **logarithmic functions** $f(x) = \log_a x$, where the **base** a is a **positive constant**.

Figure 30 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$ also $\Re$, and the function increases slowly when $x > 1$.

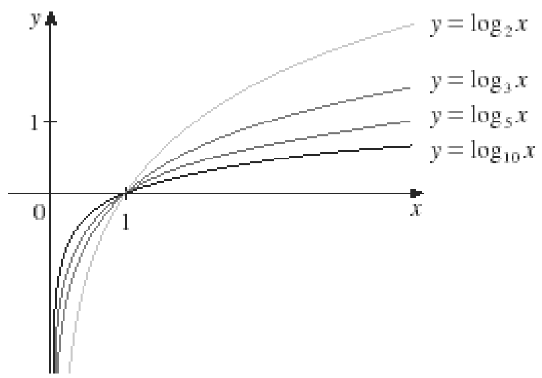


Figure 30

If $a > 0$ and $a \neq 0$, then we have

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

Thus, if $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x.

For example, $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.

We have

$$\log_a (a^x) = x \quad \text{for every } x \in \Re$$

$$a^{\log_a x} = x \quad \text{for every } x > 0$$

Laws of Logarithms

If x and y are positive numbers, then

1. $\log_a(xy) = \log_a(x) + \log_a(y)$
2. $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
3. $\log_a(x^r) = r\log_a(x)$ (*where r is any real number*)

Example 12:

Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

SOLUTION Using Law 2, we have

$$\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right) = \log_2 16 = 4$$

because $2^4 = 16$.

Natural Logarithms

The logarithm with the base e is called the **natural logarithm** and is written in the form:

$\ln x = \log_e x$

where ***e*** is a mathematical constant, which equals approximately e = **2.718281828** and is also known as Euler's number.

We have:

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$

$$\ln(e^x) = x \qquad x \in \Re$$
$$e^{\ln x} = x \qquad x > 0$$

If we set $x = 1$, *we get*
$\ln e = 1$

Example 13:

Find x if ln x = 5.

Solution:

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x$$

Therefore, $x = e^5$.

(If you have trouble working with the "ln" notation, just replace it by $\log_e$. Then the equation becomes $\log_e x = 5$; so, by the definition of logarithm, $e^5 = x$.)

Change of Base Formula For any positive number $a\ (a \neq 1)$, we have

$$\log_a x = \frac{\ln x}{\ln a}$$

3.9. Transformation of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of certain related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs. Let's first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y-coordinate is increased by the same number c). Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x). Therefore, the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted c units to the right.

(See **Figure 31**)

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

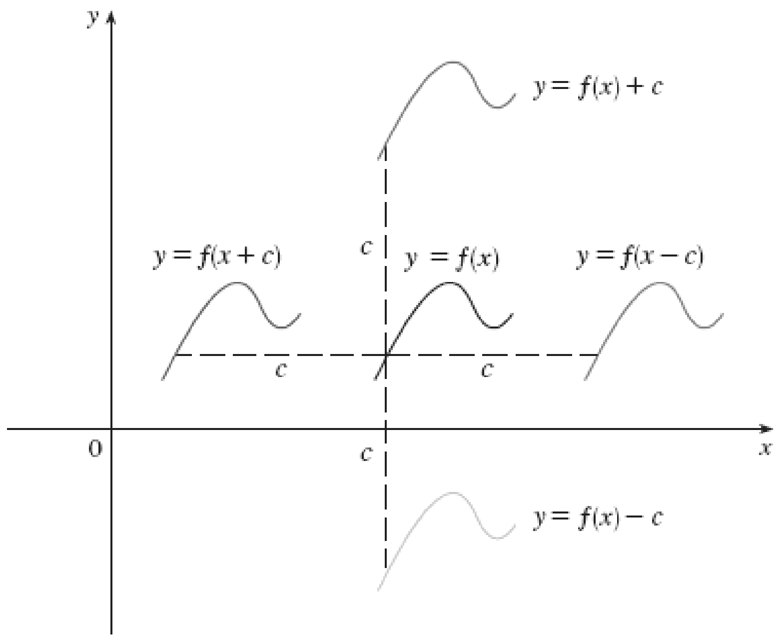


Figure 31