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On weak crossed products, Frobenius algebras, and the weak Bruhat ordering

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Abstract

The weak crossed product algebra was studied first by Haile, Larson, and Sweedler [Amer. J. Math. 105 (1983) 689]. They gave interesting properties for such an algebra, one of which is that if A_f is a weak crossed product induced by a weak 2-cocycle f defined on a Galois group $G = \mathrm{Gal}(K/F)$, and H is the inertial subgroup of G, then A_f has a Wedderburn splitting, that is $A_f = B \oplus J$ where J is the radical of A_f and B is a K^H-central simple algebra. The purpose of this paper is to give the necessary and sufficient condition for a weak crossed product to be Frobenius and to describe an algorithm for constructing lower subtractive graphs from a finite group G and a generating set S. A special case of this construction is the so-called weak Bruhat ordering on a Coxeter group (G, S). We show that the nilCoxeter algebra associated to (G, S) is a special case of the restricted algebra associated to a lower subtractive graph.

0. Introduction

Let K/F be a Galois extension of fields, $G = \mathrm{Gal}(K/F)$. A weak 2-cocycle is a function $f : G \times G \to K$ such that $f^\sigma(\tau,\gamma)f(\sigma,\tau\gamma) = f(\sigma,\tau)f(\sigma\tau,\gamma)$ and $f(1,\sigma) = f(\sigma,1) = 1$ for all $\sigma,\tau,\gamma \in G$. Notice that f can take the value $0 \in K$. In the classical case the values of f are always invertible. A K-algebra can be obtained from f in the following manner: let A_f be the K-vector space with basis $\{x_\sigma \mid \sigma \in G\}$, so $A_f = \sum_{s\in G} K x_\sigma$, and define the product in A_f by

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(1) $x_\sigma x_\tau = f(\sigma,\tau)x_{\sigma\tau}$,
(2) $k^\sigma x_\sigma = x_\sigma k$ for $k \in K$.

This algebra is called the *Weak Crossed Product* associated to f. Define a relation on G by $\sigma \leqslant \tau$ if and only if $f(\sigma^{-1},\sigma\tau) \neq 0$. The inertial subgroup H is defined by $H = \{\sigma \in G \mid \sigma \leqslant 1\}$. It has been shown that A_f can be written as a direct sum $\sum_{\sigma\in H} Kx_\sigma \oplus \sum_{\sigma\notin H} Kx_\sigma$ where the first part is a K^H-central simple algebra denoted by B and the second part is the radical of A_f denoted by J. This splitting is known as the Wedderburn splitting. On G/H we define an induced relation by $\sigma H \leqslant \tau H$ if and only if $\sigma \leqslant \tau$. This is a partial order on G/H with unique least element H, and it satisfies the following property which is called lower subtractivity: if $\sigma H \leqslant \tau H$ then

$$\sigma H \leqslant \gamma H \leqslant \tau H \quad\Longleftrightarrow\quad \sigma^{-1}\gamma H \leqslant \sigma^{-1}\tau H.$$

So, we obtain a graph Γ_f^L on G/H associated with f and rooted at H, called the left graph of f. If the original relation is changed to $\sigma \leqslant \tau$ if and only if $f(\tau\sigma^{-1},\sigma) \neq 0$, then we get the right graph on $H\backslash G$ in an analogous way. We denote the right graph of the cocycle f by Γ_f^R. To each weak 2-cocycle f is associated an idempotent weak 2-cocycle e taking the values 0 and 1, and defined by

$$e(\sigma,\tau) = 0 \quad \text{if and only if} \quad f(\sigma,\tau) = 0.$$

These two 2-cocycles have the same lower subtractive graph [4,5].

An F-algebra A is called Frobenius if there is a non-degenerate bilinear form $T : A \times A \to F$ such that $T(ab,c) = T(a,bc)$ for all $a,b,\in A$. Most references call the bilinear form associative if the later property holds. Equivalently, A is Frobenius if there is a linear map $\lambda : A \to F$ whose kernel does not contain any non-trivial one-sided ideal. Frobenius algebras satisfy the double annihilator statements which are $\ell.\mathrm{ann}(r.\mathrm{ann}(I^\ell)) = I^\ell$ and $r.\mathrm{ann}(\ell.\mathrm{ann}(I^r)) = I^r$ where I^ℓ is any left ideal and I^r is any right ideal [2].

For a Frobenius algebra A with bilinear form T, there is a unique automorphism φ of A satisfying

$$T(x,y) = T\big(\varphi(y),x\big) \quad \text{for all } x,y \in A.$$

This automorphism is called the Nakayama automorphism for T. The Frobenius algebra A with bilinear form T is called symmetric if $T(x,y) = T(y,x)$ for all $x,y \in A$, or equivalently if the Nakayama automorphism of T is inner. A standard example for a symmetric algebra is the group algebra. (For more about Frobenius algebras, see [7,8].)

This paper consists of three sections in addition to the introduction. In the first section we give the necessary and sufficient condition for the weak crossed product A_f to be Frobenius. A sub-algebra of A_e where e is an idempotent weak 2-cocycle, is the restricted algebra that we study in Section 2. In Section 3, we introduce an algorithm by which we always get a lower subtractive graph with special properties over a given finite group and generating set. A part of this section is devoted to showing a connection between the restricted algebra and the so-called nilCoxeter algebra in case of Coxeter group.

1. When is A_f Frobenius?

The Frobenius property of the weak crossed product A_f depends completely on the graph of f. We shall prove that A_f is Frobenius if and only if the graph of f has a unique maximal element.

Lemma 1.1. *Let f be a weak 2-cocycle defined on G with inertial subgroup H and let $A_f = B + J$ be the Wedderburn splitting of A_f, then*

$$A_f = \sum_{\sigma \in S} Bx_\sigma \quad (\textit{direct sum}),$$

where $\{\sigma' s\} = S$ is a set of representatives of the cosets $H\backslash G$.

Proof. Notice that

$$\begin{aligned} A_f &= \sum_{\lambda \in G} Kx_\lambda = \sum_{\sigma \in S} \sum_{h \in H} Kx_{h\sigma} \\ &= \sum_{\sigma \in S} \sum_{h \in H} Kx_h x_\sigma \quad (\text{since } h \leqslant h\sigma \quad \text{for all } h \in H, \sigma \in G) \\ &= \sum_{\sigma \in S} \left(\sum_{h \in H} Kx_h \right) x_\sigma = \sum_{\sigma \in S} Bx_\sigma. \qquad \Box \end{aligned}$$

A similar proof gives that $A_f = \sum_{\tau \in S'} x_\tau B$, where S' is a set of representatives of the coset G/H.

Lemma 1.2. *Let f be a weak 2-cocycle defined on G with inertial subgroup H. The left graph of f has a unique maximal element if and only if the right graph of f has a unique maximal element. More precisely, if $\gamma \in G$, then γH is the unique maximal element of the left graph if and only if $H\gamma$ is the unique maximal element of the right graph.*

Proof. Let $\gamma \in G$, then $H\gamma$ is the unique maximal element in $S \Leftrightarrow H\sigma \leqslant H\gamma$ for all $\sigma \in G \Leftrightarrow f(\gamma\sigma^{-1}, \sigma) \neq 0$ for all $\sigma \in G \Leftrightarrow f(\tau, \tau^{-1}\gamma) \neq 0$ for all $\tau \in G \Leftrightarrow \tau H \leqslant \gamma H$ for all $\tau \in G$ that is γH is the maximal element in the left graph on G/H. $\Box$

Let

$$P = \{\gamma \in G \mid aH \leqslant bH \Leftrightarrow \sigma a \sigma^{-1} H \leqslant \sigma b \sigma^{-1} H \text{ for all } a, b \in G\}$$

the set of all elements in G preserving the order. Observe that P is indeed a subgroup of G. Moreover, since H is the unique least element of the partial order, P is a subgroup of the normalizer of H in $G(N_G(H))$.

Proposition 1.3. *If the graph of f has a unique maximal coset γH then $\gamma \in P$.*

Proof. Observe that

$$\begin{aligned}\sigma H \leqslant \tau H \quad &\Leftrightarrow \quad \sigma H \leqslant \tau H \leqslant \gamma H \quad \Leftrightarrow \quad \sigma^{-1}\tau H \leqslant \sigma^{-1}\gamma H \leqslant \gamma H \\ &\Leftrightarrow \quad \tau^{-1}\gamma H \leqslant \tau^{-1}\sigma\gamma H \leqslant \gamma H \quad \Leftrightarrow \quad \gamma^{-1}\sigma\gamma H \leqslant \gamma^{-1}\tau\gamma H.\end{aligned}$$

So $\gamma^{-1} \in P$ and thus $\gamma \in P$. □

Corollary 1.4. *Let f be a weak 2-cocycle with inertial subgroup H. Let γH be the unique maximal coset in the graph of f. Then $Bx_\gamma = x_\gamma B$.*

Proof. Let $x_\gamma x_h \in x_\gamma B$, then $x_\gamma x_h = f(\gamma, h)x_{\gamma h}$. But by the proposition above $\gamma h = h'\gamma$ for some $h' \in H$. So

$$x_\gamma x_h = f(\gamma, h)x_{h'\gamma} = \frac{f(\gamma, h)}{f(h', \gamma)} x_{h'} x_\gamma \in Bx_\gamma. \qquad \square$$

Lemma 1.5. *Let $\gamma H, \gamma' H$ be two distinct maximal elements in the graph of f. Then $A_f x_\gamma \neq A_f x_{\gamma'}$.*

Proof. This is obvious since $A_f x_\gamma = A_f x_{\gamma'}$ would imply that $\gamma H \leqslant \gamma' H \leqslant \gamma H$ in the right partial order, or $\gamma H = \gamma' H$. □

Theorem 1.6. *Let f, G, H, S, and A_f be as above. Then A_f is a Frobenius algebra if and only if the graph of f has a unique maximal element.*

Proof. ($\Leftarrow$). We have $A_f = \sum_{\sigma \in S} Kx_\sigma$ by Lemma 1.1. Let γH be the unique maximal element in the graph of f, S and define a linear map $\lambda : A_f \to F$ by

$$\lambda\left(\sum_{\sigma \in S} b_\sigma x_\sigma\right) = \mathrm{Tr}_{L/F} \circ \mathrm{Tr}_{B/L}(b_\gamma) = \mathrm{Tr}_{B/F}(b_\gamma),$$

where $\mathrm{Tr}_{B/L}$ is the reduced trace for the element b_γ, $\mathrm{Tr}_{L/F}$ is the field extension trace, and $L = K^H = \text{center}(B)$. It is easy to check linearity. We claim that $\ker\lambda$ does not contain any non-zero one-sided ideal: let $y \in \ker\lambda - \{0\}$, we proceed to show that $yA_f \nsubseteq \ker\lambda$. Let $y = b_1 x_{\sigma_1} + \cdots + b_\ell x_{\sigma_\ell}$, $b_i \in B - \{0\}$ for all i. Hence $yx_{\sigma_1^{-1}\gamma} = b'_1 x_\gamma + b'_2 x_{\sigma_2\sigma_1^{-1}\gamma} + \cdots + b'_\ell x_{\sigma_\ell\sigma_1^{-1}\gamma}$ where the first term is non-zero. If everything else is zero we stop. Otherwise, if $b'_i x_{\sigma_i\sigma_1^{-1}\gamma}$ is the first non-zero term after $b'_1 x_\gamma$, then $yx_{\sigma_i\sigma_1^{-1}\gamma} x_{\gamma^{-1}\sigma_1\sigma_i^{-1}\gamma} = 0 + b''_i x_\gamma + \cdots$ (since $x_\gamma x_\sigma = 0$ for all $\sigma \notin H$). If any extra terms are left, we multiply by a suitable element so that we eventually get an element $z \in A_f$ satisfying $yz = bx_\gamma$, $b \in B - \{0\}$. So, $\lambda(yz) = \mathrm{Tr}_{B/F}(b)$ and because $b \neq 0$, we can find a non-zero element d in B such that $\mathrm{Tr}_{B/L}(bd) \neq 0$. Notice that by Corollary 1.4, $x_\gamma B = Bx_\gamma$, that is $bx_\gamma d' = bdx_\gamma$ for some d' in B. This implies $\lambda(yzd') = \mathrm{Tr}_{L/F} \circ \mathrm{Tr}_{B/L}(bd)$, where $\mathrm{Tr}_{B/L}(bd) \neq 0$. Let $\mathrm{Tr}_{B/L}(bd) = t \in L^*$. We can find an element t' in L^* such that $\mathrm{Tr}_{L/F}(tt') \neq 0$. Thus

$yzd't' \notin \ker\lambda$ or $yA_f \nsubseteq \ker\lambda$ and $\ker\lambda$ does not contain any non-zero right ideal. The left ideal case is similar.

($\Rightarrow$). Assume the graph of f has two maximal elements γH and $\gamma' H$. Consider the left ideal $A_f x_\gamma$. Since $x_\gamma x_\sigma = 0$ if and only if $\sigma \notin H \Rightarrow r.\mathrm{ann}(A_f x_\gamma) = J$. Notice also that $x_{\gamma'} \in \ell.\mathrm{ann}J$ since γ' is maximal $\Rightarrow x_{\gamma'} \in \ell.\mathrm{ann}(r.\mathrm{ann}(A_f x_\sigma)) \Rightarrow$ if A_f is Frobenius, $x_{\gamma'} \in A_f x_\gamma$ by the double annihilator statement. Likewise, we get $x_\gamma \in A_f x_{\gamma'}$. Thus, $A_f x_\gamma = A_f x_{\gamma'}$ which is not true by Lemma 1.5. So A_f is not Frobenius. □

The following consequences are true for any Frobenius algebra (see [2,7]). In particular, they are true for A_f where the graph of f has a unique maximal element γH.

Corollary 1.7. *Let f be a weak 2-cocycle whose graph has only one maximal element, then*:

(i) $A_f \simeq A_f^*$ *as left A_f-modules (A_f^* is the dual of A_f).*
(ii) A_f *is injective as a left A_f-module.*

Proposition 1.8. *Let f be a weak 2-cocycle having a unique maximal element γH in its graph, with inertial subgroup H, $A_f = B + J$, the Wedderburn splitting of A_f. Then*

$$\textit{Right Annihilator}\,(J) = \textit{Left Annihilator}\,(J) = A_f x_\gamma = x_\gamma A_f = \textit{Socle}\, A_f.$$

Proof. The two annihilators in the statement are 2-sided ideals. Assume that $Jx_t = 0 \Rightarrow x_\sigma x_t = 0$ for all $\sigma \notin H \Rightarrow f(\sigma, t) = 0$ for all $\sigma \notin H \Rightarrow t \notin H$ and $\sigma \nleqslant \sigma t$ for all $\sigma \notin H$. If $t \neq \gamma$, we can take $\sigma = \gamma t^{-1}$, in particular, to get $\gamma t^{-1} \nleqslant \gamma$ which contradicts the maximality of γH. Now, we show that $t = \gamma$. If $x_{\sigma'} x_\gamma \neq 0$ for some $\sigma' \notin H$, then $f(\sigma', \gamma) \neq 0$ or $\sigma' \leqslant \sigma'\gamma$ and hence $\sigma' \leqslant \sigma'\gamma \leqslant \gamma$. By lower subtractivity, $\sigma'^{-1}\sigma'\gamma \leqslant \sigma'^{-1}\gamma$ or $\gamma \leqslant \sigma'^{-1}\gamma$ which contradicts the maximality of γ. So $t = \gamma$ and $r.\mathrm{ann}J = x_\gamma A_f$. If now $x_t x_\sigma = 0$ for all $\sigma \notin H$ then $f(t, \sigma) = 0$ so $t \notin H$ and $t \nleqslant t\sigma$ for all $\sigma \notin H$. In particular, by taking $\sigma = t^{-1}\gamma$ (if $t \neq \gamma$), we get $t^{-1}\gamma \nleqslant \gamma$ (contradiction). If $t = \gamma$ then clearly $x_\gamma x_\sigma = 0$ for all $\sigma \notin H$ since γ cannot be less than $\gamma\sigma$ for any $\sigma \notin H$. Hence $\ell.\mathrm{ann}J = A_f x_\gamma$. But we showed in Corollary 1.4 that $x_\gamma A_f = A_f x_\gamma$. The last equality follows from [4, Proposition 2.5]. □

Let the group G act on the monoid $M^2(G, K)$ in the following way:

$$(\sigma * f)(a, b) = f^\sigma\big(\sigma a \sigma^{-1}, \sigma b \sigma^{-1}\big).$$

Lemma 1.9. *If $\sigma \in G$ satisfies $\sigma * f \sim f$, then σ preserves the order on the graph of f.*

Proof. If $\sigma * f \sim f$ then there exists β (weak 1-cocycle) such that

$$f(a, b) = \frac{\beta(t)\beta^t(r)}{\beta(tr)} f^\sigma\big(\sigma a \sigma^{-1}, \sigma b \sigma^{-1}\big). \tag{1.1}$$

Assume that $a \leqslant b$ and $\sigma a\sigma^{-1} \nleqslant q\sigma b\sigma^{-1} \Rightarrow f(a, a^{-1}b) \neq 0$ and $f(\sigma a\sigma^{-1}, \sigma a^{-1}b\sigma^{-1}) = 0$ which is impossible by (1.1). $\square$

Under this action, let P' be the isotropy subgroup fixing f. Lemma 1.9 shows that P' is a subgroup of P.

Proposition 1.10. *Let f be a weak 2-cocycle whose graph has a unique maximal element γH. Let λ be the linear map defined in the proof of Theorem 1.6. If φ is the Nakayama automorphism induced by λ, then $\varphi(k) = k^\gamma$ for all $k \in K$.*

Proof. For every $b \in B$, $k \in K$, we have

$$\lambda(bx_\gamma k) = \lambda\big(bk^\gamma x_\gamma\big) = \operatorname{Tr}\big(bk^\gamma\big).$$

On the other hand, $\lambda(bx_\gamma k) = \lambda(\varphi(k)bx_\gamma) = \operatorname{Tr}(\varphi(k)b)$. But the trace map satisfies $\operatorname{Tr}(bk^\gamma) = \operatorname{Tr}(k^\gamma b)$, so we get $\operatorname{Tr}(k^\gamma b) = \operatorname{Tr}(\varphi(k)b)$ for all $b \in B \Rightarrow \varphi(k) = k^\gamma$. $\square$

This statement tells us that $\varphi|_K = \gamma$. Let $\varphi(x_\sigma) = \sum k_g x_g$. So $k^\gamma\varphi(x_\sigma) = \varphi(kx_\sigma) = \varphi(x_\sigma k^{\sigma^{-1}}) = \varphi(x_\sigma)\varphi(k^{\sigma^{-1}}) \Rightarrow$

$$\sum k^\gamma k_g x_g = \sum k_g x_g k^{\gamma\sigma^{-1}} = \sum k^{g\gamma\sigma^{-1}} k_g x_g$$

$\Rightarrow \gamma = g\gamma\sigma^{-1} \Rightarrow g = \gamma\sigma\gamma^{-1}$ and $\varphi(x_\sigma) = k_\sigma x_{\gamma\sigma\gamma^{-1}}$ for some $k_\sigma \in K^*$. The equation $\varphi(x_\sigma x_\tau) = \varphi(x_\sigma)\cdot\varphi(x_\tau)$ implies that

$$\begin{aligned}
&f^\gamma(\sigma,\tau)k_{\sigma\tau}x_{\gamma\sigma\tau\gamma^{-1}} = k_\sigma k_\tau^{\gamma\sigma\gamma^{-1}} f\big(\gamma\sigma\gamma^{-1}, \gamma\tau\gamma^{-1}\big)x_{\gamma\sigma\tau\gamma^{-1}} \\
&\quad\Rightarrow \quad f^\gamma(\sigma,\tau)k_{\sigma\tau} = k_\sigma k_\tau^{\gamma\sigma\gamma^{-1}} f\big(\gamma\sigma\gamma^{-1}, \gamma\tau\gamma^{-1}\big) \\
&\quad\Rightarrow \quad f(\sigma,\tau) = \frac{k_\sigma^{\gamma^{-1}}(k_\tau^{\gamma^{-1}})^\sigma}{k_{\sigma\tau}^{\gamma^{-1}}} f^{\gamma^{-1}}\big(\gamma\sigma\gamma^{-1}, \gamma\tau\gamma^{-1}\big)
\end{aligned}$$

$\Rightarrow \gamma^{-1} * f \sim f$ via the weak 1-cocycle β which is given by $\beta(\sigma) = k_\sigma^{\gamma^{-1}}$. This shows that the maximal element γ must belong to the subgroup P' (the isotropy subgroup) defined after Lemma 1.9.

To find k_σ, notice that

$$\begin{aligned}
T(x_{\gamma\sigma^{-1}}, kx_\sigma) &= T\big(k^{\gamma\sigma^{-1}}x_{\gamma\sigma^{-1}}, x_\sigma\big) = \lambda\big(k^{\gamma\sigma^{-1}} f\big(\gamma\sigma^{-1},\sigma\big)x_\gamma\big) \\
&= \operatorname{Tr}\big(k^{\gamma\sigma^{-1}} f\big(\gamma\sigma^{-1},\sigma\big)\big) \\
&= \operatorname{Tr}\big(\big(k^{\gamma\sigma^{-1}}\big)^{\gamma\sigma\gamma^{-1}} f^{\gamma\sigma\gamma^{-1}}\big(\gamma\sigma^{-1},\sigma\big)\big) \quad \big(\text{since } T_r(k) = T_r\big(k^\sigma\big)\big) \\
&= \operatorname{Tr}\big(k^\gamma f^{\gamma\sigma\gamma^{-1}}\big(\gamma\sigma^{-1},\sigma\big)\big).
\end{aligned}$$

On the other hand,

$$\begin{aligned} T(x_{\gamma\sigma^{-1}}, kx_\sigma) &= T(\varphi(kx_\sigma), x_{\gamma\sigma^{-1}}) = T\big(k^\gamma k_\sigma x_{\gamma\sigma\gamma^{-1}}, x_{\gamma\sigma^{-1}}\big) \\ &= \mathrm{Tr}\big(k^\gamma k_\sigma f\big(\gamma\sigma\gamma^{-1}, \gamma\sigma^{-1}\big)\big) \\ &\Rightarrow \mathrm{Tr}\big(k^\gamma f^{\gamma\sigma\gamma^{-1}}\big(\gamma\sigma^{-1}, \sigma\big)\big) = \mathrm{Tr}\big(k^\gamma k_\sigma f\big(\gamma\sigma\gamma^{-1}, \gamma\sigma^{-1}\big)\big) \\ &\qquad \text{for all } k \in K \\ &\Rightarrow k_\sigma = \frac{f^{\gamma\sigma\gamma^{-1}}(\gamma\sigma^{-1}, \sigma)}{f(\gamma\sigma\gamma^{-1}, \gamma\sigma^{-1})}. \end{aligned} \tag{1.2}$$

We have now all the ingredients to write down the Nakayama automorphism for our linear map λ explicitly:

$$\varphi_\lambda(kx_\sigma) = k^\gamma \frac{f^{\gamma\sigma\gamma^{-1}}(\gamma\sigma^{-1}, \sigma)}{f(\gamma\sigma\gamma^{-1}, \gamma\sigma^{-1})} x_{\gamma\sigma\gamma^{-1}}.$$

Here are some consequences of this derivation.

Corollary 1.11. *Let f be a weak 2-cocycle whose graph has a unique maximal element γH. Then $\gamma \in P'$. Moreover, if b is any invertible element in A_f, then*

$$\psi(kx_\sigma) = \varphi_{b\lambda}(kx_\sigma) b^{-1} k^\gamma \frac{f^{\gamma\sigma\gamma^{-1}}(\gamma\sigma^{-1}, \sigma)}{f(\gamma\sigma\gamma^{-1}, \gamma\sigma^{-1})} x_{\gamma\sigma\gamma^{-1}} b \tag{1.3}$$

is a Nakayama automorphism for some associative non-degenerate bilinear form on A_f.

Corollary 1.12. *Let f be a graph with a unique maximal element γH, $H \neq G$, k_σ as given in* (1.2). *Let $\psi : A_f \to A_f$ be a Nakayama F-automorphism. Then*

(i) $\psi \notin \mathrm{Aut}_K A_f$.
(ii) ψ *is not inner.*
(iii) $\psi(K) = K$ *if and only if* $\psi = \varphi_{\ell_h x_h \lambda}$ *for some* $h \in H$, $\ell_h \in K^*$, *and in this case* $\psi|_K = \gamma h$, $h \in H$.

Proof. (i) If $\psi \in Aut_K A_f$ then $\psi(k) = k$, but by the proposition above $\psi(k) = b^{-1}k^\gamma b$ for some invertible element $b \in A_f \Rightarrow b^{-1}k^\gamma b = k$ for all $k \in K$ or $bk = k^\gamma b \Rightarrow b = tx_\gamma$ for some $t \in K$, which is impossible since x_γ is not invertible.

(ii) If ψ is inner $\Rightarrow \psi(k) = c^{-1}kc$ for some invertible element $c \in A_f$, but by the proposition above, $\psi(k) = b^{-1}k^\gamma b$ for some invertible element $b \in A_f \Rightarrow b^{-1}k^\gamma b = c^{-1}kc$ for all $k \in K$ or $k^\gamma bc^{-1} = bc^{-1}k$ for all $k \in K \Rightarrow bc^{-1} = tx_\gamma$ for some $t \in K$ (contradiction) as in (i).

(iii) Let $\psi|_K = \tau \Rightarrow \psi(k) = k^\tau =$ (by the proposition above) $b^{-1}k^\gamma b$ for some invertible element $b \in A_f \Rightarrow bk^\tau = k^\gamma b$ for all $k \in K \Rightarrow b =$ for some $t \in k$ $tx_{\gamma\tau^{-1}} \Rightarrow$

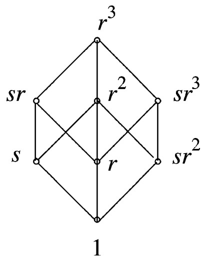

Fig. 1.

$\tau \in H\gamma = \gamma H$ $b = \ell_h x_h$, for some $\ell_h \in K^*$, $h \in H$. Conversely, if ψ is given by the formula (1.3) with $b = \ell_h x_h$ then $\psi(k) = k^{h\gamma} \Rightarrow \psi(K) = K$. □

Remark 1.13. On the set of all Nakayama automorphisms, we define an equivalence relation by $\psi_1 \sim \psi_2$ if and only if $\psi_1(a) = k^{-1}\psi_2(a)k$, for some $k \in K^*$. Then, Corollary 1.12(iii) establishes a 1-1 correspondence between the set of all equivalence classes and the coset γH. In particular, if $H = \{1\}$ then φ is unique up to conjugation by a non-zero element of K.

Example 1.1. The graph (Fig. 1) on the Dihedral group of order 8, $D_4 = \langle s, r \mid s^2 = srsr = r^4 = 1\rangle$, which is arising from an idempotent weak 2-cocycle e is lower subtractive, and has a unique maximal vertex r^3. If f is a weak 2-cocycle associated to e, then $A_f = \sum_{\sigma \in D_4} K x_\sigma$ is Frobenius, and the Nakayama automorphism which takes K to itself can be given by

$$\varphi(kx_{sr^j}) = k^{r^3} \frac{f^{sr^j}(sr^{j-3}, sr^j)}{f(sr^{j-2}, sr^{j-3})} x_{sr^{j-2}} \quad \text{and} \quad \varphi(kx_{r^j}) = k^{r^3} \frac{f^{r^j}(r^{3-j}, r^j)}{f(r^j, r^{3-j})} x_{r^j}.$$

Proposition 1.14. *Let f be a weak 2-cocycle whose graph has a unique maximal element γH, where H is the inertial subgroup. The algebra A_f is symmetric if and only if $H = G$.*

Proof. ($\Rightarrow$). If $H \neq G$ then we have seen that the Nakayama automorphism can be given by $\varphi(kx_\sigma) = k^\gamma k_\sigma x_{\gamma\sigma\gamma^{-1}}$ and by Corollary 1.12(ii) φ is not inner so A_f is not symmetric.
($\Leftarrow$). If $H = G$ then A_f is a central simple F-algebra and therefore is symmetric. □

2. The restricted subalgebra $\overline{A}_e$

With inessential changes, one can follow the proof of Theorem 1.6 and show that if T is a subgroup of G and f is a weak 2-cocycle satisfying

$$f(\sigma, \tau) \in K^T \quad \text{for all } \sigma, \tau \in G, \tag{2.1}$$

then the graph of f has a unique maximal element if and only if $\overline{A}_f = \sum_{\sigma\in G} K^T x_\sigma$ is Frobenius. We insist to have condition (2.1) held because otherwise $\overline{A}_f$ would not be a K^T-algebra. In particular, if we consider an idempotent weak 2-cocycle e whose values lie in $\{0,1\} \subset F$ and define $\overline{A}_e := \sum_{\sigma\in G} F x_\sigma$, the restricted algebra then we have the following corollary.

Corollary 2.1. *Let e be an idempotent weak* 2*-cocycle. The graph of e has a unique maximal element γH if and only if $\overline{A}_e$ is Frobenius.*

Proof. As we saw before,

$$\overline{A_e} = \sum_{h\in H} F x_{\sigma_h} \oplus \sum_{\sigma\notin H} F x_\sigma = FH \oplus \sum_{\sigma\notin H} F x_\sigma = \sum_{\sigma\in S} FH x_\sigma.$$

Now for any element $a \in \overline{A}_e$, $a = \sum_{\sigma\in S} b_\sigma x_\sigma$, $b_\sigma \in FH$. Define a linear map $\bar\lambda : \overline{A}_e \longrightarrow F$ by

$$\bar\lambda(a) = \alpha_1 \quad \text{where } b_\gamma = \sum \alpha_h x_h.$$

If $\ker\bar\lambda$ contains a left ideal I and $a \in I - \{0\}$, then as we did in the proof of Theorem 1.6 we can find an element $y \in \overline{A}_e$ such that $ya = b_\gamma x_\gamma$, $b_\gamma = \sum_{i=1}^r \alpha_i x_{h_i}$, $\alpha_i \neq 0$, so $\bar\lambda(x_{h_1^{-1}} ya) \neq 0$. The other direction can be shown exactly as we did in the proof of Theorem 1.6. □

Theorem 2.2. *Let e be an idempotent whose graph has a unique maximal element γH, H the inertial subgroup. Then $\overline{A}_e$ is symmetric if and only if there exists $h_0 \in H$ with $\gamma h_0 \in \operatorname{center} G$.*

Proof. ($\Leftarrow$). Assume that $\gamma h_0 \in \operatorname{center}(G)$ for some $h_0 \in H$, then we choose γh_0 to be a representative for γH in the definition of $\bar\lambda$ above and define

$$\bar\lambda\left(\sum_{\sigma\in S} b_\sigma x_\sigma\right) = \alpha_1 \quad \text{where } b_{\gamma h_0} = \sum_{h\in H} \alpha_h h.$$

Let $a = \sum_{\sigma\in G} p_\sigma x_\sigma$, $b = \sum_{\sigma\in G} q_\sigma x_\sigma$, $p_\sigma, q_\sigma \in F$. The coefficient of $x_{\gamma h_0}$ in the product ab:

$$(ab)_{\gamma h_0} = \sum_{\sigma\in G} p_\sigma q_{\sigma^1 \gamma h_0}, \quad \text{and similarly}$$

$$(ba)_{\gamma h_0} = \sum_{\sigma\in G} q_\sigma p_{\sigma^{-1}\gamma h_0} = \sum_{\tau\in G} p_\tau q_{\gamma h_0 \tau^{-1}} = \sum_{\sigma\in G} p_\sigma q_{\sigma^{-1}\gamma h_0}.$$

(⇒). Assume $\gamma h \notin \operatorname{center}(G)$ for all $h \in H$. So for each $h \in H$ there exists $\sigma_h \in G$ such that $\gamma h \sigma_h \neq \sigma_h \gamma h$. We may pick the representative γh_0 in the definition of $\bar{\lambda}$ to be γh. Consider the elements

$$a = x_{\sigma_h}, \qquad b = k_{\sigma_h^{-1}\gamma h} x_{\sigma_h^{-1}\gamma h} + k_{\gamma h \sigma_h^{-1}} x_{\gamma h \sigma_h^{-1}},$$

where $k_{\sigma_h^{-1}\gamma h} \neq k_{\gamma h \sigma_h^{-1}}$, elements from F. So

$$(ab)_{\gamma h} = k_{\sigma_h^{-1}\gamma h} \neq k_{\gamma h \sigma_h^{-1}} = (ba)_{\gamma h}.$$

Thus $\bar{\lambda}(ab) \neq \bar{\lambda}(ba)$ and $\overline{A}_e$ is not symmetric. □

Remark 2.3. The results remain true if we use the order on the right graph instead.

Corollary 2.4. *The restricted algebra $\overline{A}_e$, for an idempotent e whose graph has unique maximal element, is symmetric if G is Abelian.*

Let A be Frobenius containing an ideal I. Under which circumstances is A/I Frobenius? Jans answered this question in general in his paper [7]. He showed that A/I is Frobenius if and only if there exists an element $c \in A$ such that $r.\mathrm{ann}(I) = Ac = cA$, and in this case a non-degenerate associative bilinear form $\overline{T}$ can be given on A/I by

$$\overline{T}(a + I, b + I) = T(ab, c). \tag{2.2}$$

Suppose the inertial subgroup is trivial and let I be two-sided ideal in A_f, then A_f/I is Frobenius if and only if $r.\mathrm{ann}(I) = cA_f = A_f c$ for some $c = \sum_{i=1}^n k_i x_{\sigma_i} \in A_f$. Since A_f is Frobenius, this gives $I = \ell.\mathrm{ann}(cA_f)$ as right ideals. So $x_\tau \in I$ if and only if x_τ annihilates all x_{σ_i}'s from the left or $x_\tau \in \bigcap_{i=1}^n \ell.\mathrm{ann}(x_{\sigma_i} A_f)$. Thus

$$I = \bigl\langle \{x_\tau \mid \sigma_i \nleqslant_R \tau\sigma_i \text{ for all } i = 1, \ldots, n\} \bigr\rangle = \bigcap_{i=1}^n \ell.\mathrm{ann}(x_{\sigma_i} A_f).$$

Define the stable set in G, $\mathrm{St}(G)$ to be the set $\{\sigma \in G \mid \sigma \leqslant_L \tau$ if and only if $\sigma \leqslant_R \tau$ for all $\tau \in G\}$ where $\leqslant_L$ is the order in the left graph and $\leqslant_R$ is the order in the right graph. Clearly $x_\sigma A_f = A_f x_\sigma$ if and only if $\sigma \in \mathrm{St}(G)$. We formulate the following proposition.

Proposition 2.5. *If f is a weak 2-cocycle whose graph has a unique maximal vertex γ, let I be a two-sided ideal in A, then A_f/I is Frobenius if and only if*

$$I = \ell.\mathrm{ann}\left(\left(\sum_{i=1}^n k_i x_{\sigma_i}\right) A_f\right),$$

where $\sigma_i \in \mathrm{St}(G)$, $G = \mathrm{Gal}(K/F)$ and $k_i \in K$.

Example 2.1. In Example 1.1, one can find the right graph and obtain that $\mathrm{St}(D_4) = \{r, r^2, r^3, sr, sr^3\}$. So A_f/I is Frobenius if and only if $I = \ell.\mathrm{ann}((\sum_{\sigma \in V} k_\sigma x_\sigma)A_f)$ where $k_\sigma \in K$, $V \subseteq \mathrm{St}(D_4)$. In particular, if $I_0 = \ell.\mathrm{ann}(x_r A_f) = \langle x_{sr}, x_{sr^3}, x_{r^3}\rangle$, then A_f/I_0 is Frobenius.

Corollary 2.6. *In the setting above, if the left and right graphs are identical, then A_f/I is Frobenius for each ideal I.*

Proof. In this case we have $cA_f = A_f c$ for all $c \in A_f$. For a given ideal I, let $r.\mathrm{ann} I = \langle x_{\tau_i}\rangle$, then, $I = \ell.\mathrm{ann}((\sum x_{\tau_i})A_f)$ where $\sum x_{\tau_i} A_f = A_f \sum x_{\tau_i}$ and A_f/I is Frobenius. □

3. An application

From now on, we assume the inertial subgroup is trivial.

3.1. Standard graph and Bruhat ordering

Let (G, S) be a group with a finite generating set $S = \{s_1, s_2, \ldots, s_k\}$. Here, we suppose that S generates G as a semigroup which means every element in G can be expressed as a product of some elements in S without using the inverses of the elements of S. There is a natural way to construct a lower subtractive graph on G in the following manner: put $1 \in G$ in the zero level, that is, put 1 as a unique root of the graph. In the first level, right above 1, put all generators $s_1, s_2, \ldots, s_k$. For level 2, put $s_1^2, s_1 s_2, \ldots, s_1 s_k$ above s_1 unless $s_1 s_i$ has already appeared in a lower level. The elements $s_2 s_1, s_2^2, s_2 s_3, \ldots, s_2 s_k$ should be put above s_2 unless $s_2 s_i$ appeared in level 0 or 1. Repeat the same thing to put elements above $s_3, s_4, \ldots, s_k$. To construct the third level, suppose that g is in the second level, then $gs_1, gs_2, \ldots, gs_k$ which have not appeared yet in a lower level should be put right above g. All elements gs_i of level less than 3 should be ignored. Continue in this process until the elements of G are all exhausted. A graph constructed in this way will be called *The Left Standard Graph associated to* (G, S), *or* $\mathrm{LSG}(G, S)$. The right standard graph associated to (G, S) (the $\mathrm{RSG}(G, S)$) is defined in a similar way. We give a simple example.

Example 3.1. Consider the Abelian group $(G, S) = (\mathbb{Z}_n \times \mathbb{Z}_m = \langle s\rangle \times \langle t\rangle, S = \{s, t\})$. The $\mathrm{LSG}(G, S)$ is shown in Fig. 2.

Definition 3.1. In the $\mathrm{LSG}(G, S)$, we define the length of $g \in G$ by

$$\ell(g) = \min\{n \mid g = s_{i_1} s_{i_2} \cdots s_{i_n},\ s_{i_j} \in S\}.$$

From the definition of the construction, if $g, g' \in G$ and there are two maximal chains from g to g' in the graph, then this means $g' = gs_{i_1} s_{i_2} \cdots s_{i_r}$ and $g' = gs_{j_1} s_{j_2} \cdots s_{j_t}$. Now if $r < t$, $\ell(g') < \ell(g) + t$ and g' would have appeared in a lower level. Similarly, we get $\ell(g') < \ell(g) + r$ if $t < r$. So $t = r$ and we get that all maximal chains from g to g' have

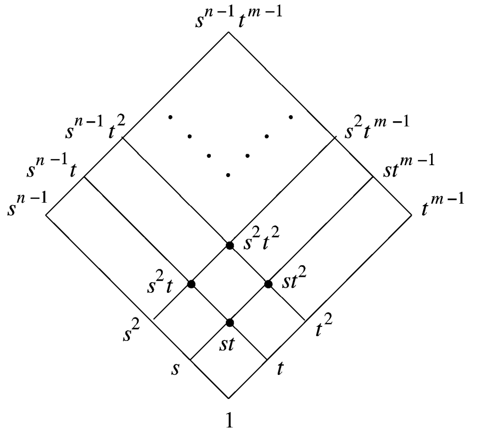

Fig. 2.

the same length. This property is called the *Catenary property*, and we have shown the following proposition.

Proposition 3.1. *The* LSG(G, S) *has the Catenary property.*

If in the LSG(G, S), g' appears in a level higher than the level of g and g' can be written as $g' = gs_{i_1}s_{i_2}\cdots s_{i_r}$ for some generators: $s_{i_1}, s_{i_2}, \ldots, s_{i_r}$, then we write $g \leqslant g'$.

Proposition 3.2. *Consider the* LSG(G, S). *Then for* $g, g' \in G$:

$$g \leqslant gg' \quad \textit{if and only if} \quad \ell(gg') = \ell(g) + \ell(g').$$

Proof. Let $\ell(g) = i$, $\ell(g') = j$, $g = s_1s_2\cdots s_i$, $g' = s_1's_2'\cdots s_j'$ (reduced expressions).

(⇒). Since gg' lies above g in the LSG, let r be the smallest integer such that $gg' = s_1s_2\cdots s_is_1''s_2''\cdots s_r''$ (reduced expression). We always have $\ell(gg') \leqslant \ell(g) + \ell(g')$ so $r \leqslant j$. On the other hand, if $r < j$ then $\ell(g')$ would be less than j which contradicts the assumption.

(⇐). Let $\ell(gg') = \ell(g) + \ell(g')$, so, $gg' = s_1s_2\cdots s_is_1's_2'\cdots s_j'$ is a reduced expression for gg'. This means gg' lies above g in the LSG or $g \leqslant_L gg'$. □

Corollary 3.3. LSG(G, S) *is lower subtractive.*

Proof. Suppose $a \leqslant b$, we need to show that

$$a \leqslant c \leqslant b \quad \Leftrightarrow \quad a^{-1}c \leqslant a^{-1}b.$$

(⇒). Since $a \leqslant b$, $a \leqslant c$, and $c \leqslant b$, we have

$$\ell(b) = \ell(a) + \ell\big(a^{-1}b\big), \tag{3.1}$$

$$\ell(c) = \ell(a) + \ell\big(a^{-1}c\big), \tag{3.2}$$

$$\ell(b) = \ell(c) + \ell\big(c^{-1}b\big). \tag{3.3}$$

Subtract (3.2) from (3.1) and then subtract (3.3) from the resulting equation to get

$$\ell\big(a^{-1}b\big) = \ell\big(a^{-1}c\big) + \ell\big(c^{-1}b\big). \tag{3.4}$$

Therefore $a^{-1}c \leqslant a^{-1}b$.

($\Leftarrow$). Let (3.1) and (3.4) be given. It is true in general that

$$\begin{aligned} &\ell(b) \leqslant \ell(c) + \ell\big(c^{-1}b\big) \\ &\quad\Rightarrow \quad \big(\text{by (3.1)}\big) \quad \ell(a) + \ell\big(a^{-1}b\big) \leqslant \ell(c) + \ell\big(c^{-1}b\big) \\ &\quad\Rightarrow \quad \big(\text{by (3.4)}\big) \quad \ell(a) + \ell\big(a^{-1}c\big) + \ell\big(c^{-1}b\big) \leqslant \ell(c) + \ell\big(c^{-1}b\big). \end{aligned}$$

So $\ell(a) + \ell(a^{-1}c) \leqslant \ell(c)$. But we always have $\ell(c) \leqslant \ell(a) + \ell(a^{-1}c)$, hence the equality holds and (3.2) follows. For (3.3), we have from (3.1) that $\ell(b) = \ell(a) + \ell(a^{-1}b) \Rightarrow$ (by (3.2) and (3.4)):

$$\ell(b) = \ell(c) - \ell\big(a^{-1}c\big) + \ell\big(a^{-1}c\big) + \ell\big(c^{-1}b\big).$$

So $\ell(b) = \ell(c) + \ell(c^{-1}b)$ which is (3.3). □

The LSG(G, S) appears in the theory of Coxeter group. We give some basic definitions and results which are well-known and can be found in [6].

Definition 3.2. A group G with generating set $S = \{s_1, s_2, \ldots, s_k\}$ is called a Coxeter group if $G = \langle s_i \in S \mid (s_i s_j)^{m_{ij}} = 1\rangle$, $m_{ij} \geqslant 2$, $i \neq j$, and $m_{ii} = 1$ for all $1 \leqslant i \leqslant k$.

In case (G, S) is a Coxeter group, the well-known Weak Bruhat Ordering by definition is exactly the LSG(G, S) in our terminology, where The Bruhat Ordering is constructed in the same method as in the LSG(G, S) except we allow multiplication from the left by not only generators but conjugates of generators.

In the Bruhat ordering, since $gys_iy^{-1} = gys_i(gy)^{-1}g$ for any conjugate ys_iy^{-1} of s_i, we could have defined the Bruhat ordering by multiplying the conjugates from the left. This property does not hold for the weak Bruhat ordering and we get two weak graphs according to which side we multiply the generators from. Multiplying the generators from the right (left) gives the left standard graph—LSG (right standard graph—RSG) which is precisely Γ_L (Γ_R). The union of these two weak Bruhat orderings in the sense that $\sigma \leqslant_U \tau$ if and only if $\sigma \leqslant x_1 \leqslant \cdots \leqslant x_i \leqslant \tau$ where $\leqslant$ is either in LSG or RSG, is a subgraph of the Bruhat ordering. In the case that $S = \{s, t\}$ with order $(st) = m$ and $st \neq ts$, we get the Dihedral group of order $2m$, and in this case the union of the LSG and RGS is identical with the Bruhat ordering. In [1], it was shown that the weak Bruhat graph is a lattice and it has a unique maximal element of order 2. So, we have the following corollary.

Corollary 3.4. *If e is the idempotent arising from the weak Bruhat ordering on a Coxeter group (G, S) and f is any cocycle on G with associated idempotent e, then A_f is Frobenius.*

Let "$\leqslant_B$" be the Bruhat ordering on a Coxeter group (G, S). Let F be a field and let $T = \{a_g, b_g \mid g \in G\} \subseteq F$, $a_1 = 1$, $a_g = a_{s_1} a_{s_2} \cdots a_{s_i}$, $b_g = b_{s_1} b_{s_2} \cdots b_{s_i}$ where $g = s_1 s_2 \cdots s_i$ is any reduced expression for g. Define an F-algebra A_T to be the free module on $\{x_g \mid g \in G\}$ and the product is given by

$$x_s x_g = \begin{cases} x_{sg} & \text{if } g \leqslant_B sg, \\ a_s x_{sg} + b_s x_g & \text{if } g \ {}_B\!\geqslant sg. \end{cases}$$

If we take $a_s = b_s = 0$ for all s, we get the so-called nilCoxeter algebra A_0. It can be characterized by

$$x_g x_{g'} = \begin{cases} x_{gg'} & \text{if } \ell(g) + \ell(g') = \ell(gg'), \\ 0 & \text{otherwise}. \end{cases}$$

See [3]. Notice that the length (level) in the LSG is the same as the length in the Bruhat ordering.

Theorem 3.5. *The nilCoxeter algebra A_0 is isomorphic to the restricted algebra $\overline{A}_e$ as F-algebras.*

Proof. Proposition 3.2 shows that $x_g x_g = x_{gg'}$ in the nilCoxeter algebra A_0 if and only if $x_g x_{g'} = x_{gg'}$ in the restricted algebra $\overline{A}_e$ where the idempotent weak 2-cocycle e is given by the weak Bruhat ordering (LSG) and hence they are isomorphic. □

Corollary 3.6. *The nilCoxeter algebra is Frobenius. Moreover, it is symmetric if and only if the maximal element in its graph belongs to the center of the Coxeter group.*

Based on Theorem 3.5, one can generalize the notion of nilCoxeter algebra to be the same as the restricted algebra $\overline{A}_e$ for a general standard lower subtractive graph.

References

[1] A. Bjorner, Orderings of Coxeter Groups, Combinatorics and Algebra, in: Contemp. Math., vol. 34, Amer. Math. Soc., Providence, RI, 1984, pp. 175–195.

[2] C. Curtis, I. Reiner, Methods of Representation Theory, vol. 1, Wiley, New York, 1981.

[3] S. Fomin, R. Stanley, Schubert polynomials and the nilCoxeter algebra, Adv. Math. 103 (1994) 196–207.

[4] D. Haile, On crossed product algebras arising from weak cocycles, J. Algebra 74 (1982) 270–279.

[5] D. Haile, R. Larson, M. Sweedler, Almost invertible cohomology theory and the classification of idempotent cohomology classes and algebras by partially ordered sets with a Galois group action, Amer. J. Math. 105 (1983) 689–814.

[6] J. Humphreys, Reflection Groups and Coxeter Groups, in: Cambridge Stud. Adv. Math., vol. 29, Cambridge Univ. Press, Cambridge, UK, 1990.
[7] J. Jans, On Frobenius algebras, Ann. of Math. 69 (1959) 392–407.
[8] K. Yamagata, Frobenius Algebras, in: Handbook of Algebra, vol. 1, 1996.