CHAPTER 6

The Critical Equation

1. Review
2. Neutron Reactions
3. Nuclear Fission
4. Thermal Neutrons
5. Nuclear Chain Reaction
6. **Neutron Diffusion**
7. *Critical Equation*

NEWS

- Final exam:

June 17th , 2008: Girls

June 12th, 2008: Boys

Lecture content:

- Introduction
- Diffusion Equation Applied to a Thermal Reactor
- Thermal Neutron Source as Obtained from the Fermi Age Equation
- Critical Equation and Reactor Buckling
- The non-leakage Factors
- Criticality of Large Thermal Reactors
- The Critical Equation for Reactors with Hydrogeneous Moderators
- Critical Size and Geometrical Buckling
- Extrapolation Length Correction
- Effect of Reflector

7.1 Introduction

Remember that: the thermal neutron flux is maximum near the center of the nuclear reactor and that it decreases as one approaches the boundaries of the assembly. Figure 7.21, page 226

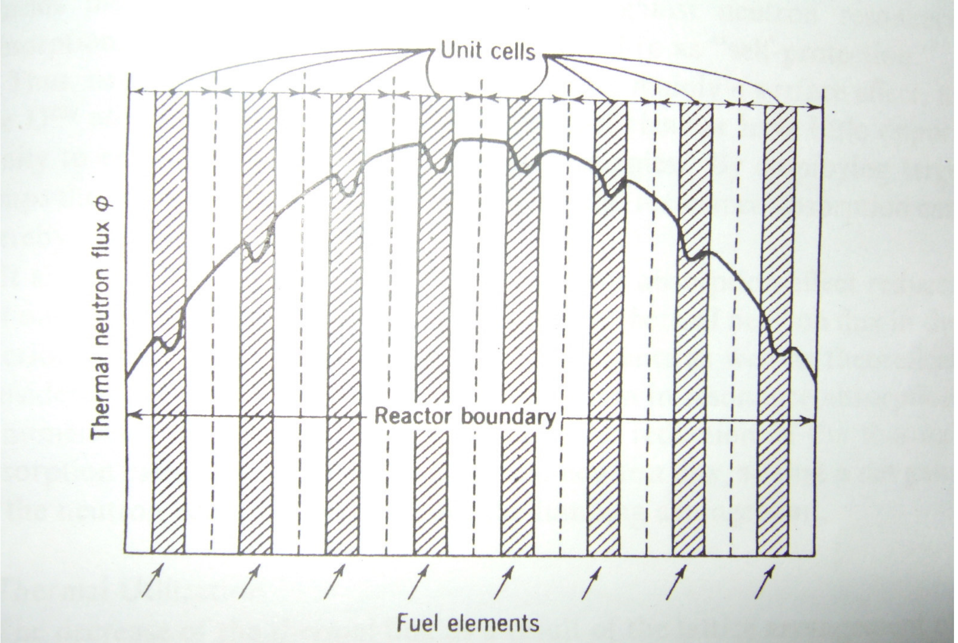

FIG. 7.21. Flux variation across reactor.

7.1 Introduction

Remember that: the thermal neutron flux is maximum near the center of the nuclear reactor and that it decreases as one approaches the boundaries of the assembly.

↓

Similar behavior for fast neutron flux : Why??

↓

Because **at the boundary**, the **diminished thermal neutron flux** leads to a **reduced fission rate** → reduced production rate of fast neutrons

↓

There will be a steady diffusion of neutrons, both fast and thermal, from the center of the reactor toward its boundaries where they will escape and be lost to the nuclear assembly

Neutron leakage

7.2 Diffusion Equation Applied to a Thermal Reactor

Remember that: Thermal neutron steady state equation is

$$\nabla^2\phi - \frac{3}{\lambda_{tr}\lambda_a}\phi + \frac{3q}{\lambda_{tr}} = 0 \quad \textbf{(7.1)}$$

Source of thermal neutrons? **from** slowing-down density q

It is a function of the space coordinates ***r*** and the neutron age τ

$$q(r)\alpha\frac{e^{-r^2/4\tau}}{\tau^{3/2}}$$

Hence the rate of supply of thermal neutrons at a given location in the reactor is determined by the value q for a neutron age $\tau = \tau_{thermal} = \tau_0$

7.2 Diffusion Equation Applied to a Thermal Reactor

Hence the first step must be to find the value of $q_{(thermal)}$ to be used in $\nabla^2\phi - \frac{3}{\lambda_{tr}\lambda_a}\phi + \frac{3q}{\lambda_{tr}} = 0$

q must first be found for thermal energies by solving the age equation $\nabla^2 q - \frac{\partial q}{\partial \tau} = 0$

7.3 Thermal Neutron Source as Obtained from the Fermi Age Equation

Assumption: **the desired solution can be written in the form of a product of two functions**

$R(\boldsymbol{r})$ and $T(\tau)$

function R depends on the space coordinates $\boldsymbol{r}$ only and the function T depends on τ only

$$q(r,t) = R(r)T(\tau)$$

Derivative with respect to space: $\nabla^2 q = T(\tau)\nabla^2 R(r)$

Derivative with respect to the Fermi age: $\dfrac{\partial q}{\partial \tau} = R(r)\dfrac{\partial T}{\partial \tau}$

$$T(\tau)\nabla^2 R(r) = R(r)\frac{\partial T}{\partial \tau} \quad \Rightarrow \quad \frac{\nabla^2 R(r)}{R(r)} = \frac{1}{T(\tau)}\frac{\partial T}{\partial \tau} \qquad \textbf{(7.4)}$$

Note that: each side of the equation is independent from the variables of the other side

since they are equal ⇒ each side must be equal to the same constant

7.3 Thermal Neutron Source as Obtained from the Fermi Age Equation

each side must be equal to the same constant

Let this constant be called $-B^2$

$$\frac{\nabla^2 R(r)}{R(r)} = -B^2 \Rightarrow \nabla^2 R(r) + B^2 R(r) = 0$$

$$\frac{1}{T(\tau)}\frac{\partial T}{\partial \tau} = -B^2 \quad \textbf{(9.5)} \xrightarrow{\text{solution}} T = T_0 \exp(-B^2 \cdot \tau)$$

Where T_0 is the value of T initially, when $\tau = 0$

Since q decreases with increasing age because of neutron losses $\Rightarrow T < T_0$

$\Downarrow$

B^2 must be a real and positive number

7.3 Thermal Neutron Source as Obtained from the Fermi Age Equation

The slowing-down density at the beginning of the slowing-down process, q_0, is then

$$q_0 = R(r)T(0) = R(r)T_0$$

The number of neutrons per cubic centimeter per second that become available for slowing down is given by the rate of production of fission neutrons

This rate is equal to $\varepsilon f \eta$ per thermal neutron absorbed

The rate of thermal neutron absorption per cubic centimeter of the reactor is $\phi \Sigma_a$

7.3 Thermal Neutron Source as Obtained from the Fermi Age Equation

the rate of production of fission neutrons per cubic centimeter is:

$$rate\ of\ \ production = \varepsilon f \eta \cdot \phi \Sigma_a$$

This is also the rate per cubic centimeter at which fast neutrons become available for slowing down, which is the same as the initial slowing-down density q_0

$$q_0 = \phi(r)\Sigma_a \varepsilon f \eta$$

since

$$q = T(\tau)R(r)$$
$$= T_0 \exp(-B^2\tau)R(r)$$
$$= q_0 \exp(-B^2\tau)$$

$$q = \phi \Sigma_a \varepsilon f \eta e^{-B^2\tau}$$

This is the desired solution of the Fermi age equation, neglecting any neutron absorption during the prethermal stage

7.3 Thermal Neutron Source as Obtained from the Fermi Age Equation

However, since we have a certain amount of **neutrons absorption** during slowing down if we use uranium, we must allow for this absorption in our calculation

Previous chapter: Multiplication by factor *p*

$$q' = qp = p\phi\Sigma_a \varepsilon f \eta e^{-B^2\tau} = k_\infty \Sigma_a \phi e^{-B^2\tau}$$

the steady state equation for thermal neutron become

$$\nabla^2\phi - \frac{3}{\lambda_{tr}\lambda_a}\phi + \frac{3k_\infty e^{-B^2\tau}}{\lambda_{tr}\lambda_a}\phi = 0$$

Using: $L^2 = \frac{1}{3}\lambda_{tr}\lambda_a$

$$\nabla^2\phi + \frac{k_\infty e^{-B^2\tau} - 1}{L^2}\phi = 0$$

7.4 Critical Equation and Reactor Buckling

$$q' = qp = p\phi\Sigma_a \varepsilon f \eta e^{-B^2\tau} = k_\infty \Sigma_a \phi e^{-B^2\tau}$$

⇓

$$\nabla^2 q' = k_\infty \Sigma_a e^{-B^2\tau} \nabla^2 \phi$$

⇓

$$\frac{\partial q'}{\partial \tau} = k_\infty \Sigma_a \phi (-B^2) e^{-B^2\tau}$$

⇓

$$\nabla^2 q' - \frac{\partial q'}{\partial \tau} = k_\infty \Sigma_a e^{-B^2\tau} (\nabla^2 \phi + B^2 \phi) = 0$$

⇓

$$\nabla^2 \phi + B^2 \phi = 0$$

7.4 Critical Equation and Reactor Buckling

$$\nabla^2\phi + B^2\phi = 0$$

B^2 is called **geometrical buckling** because of its intimate connection to the geometry of the nuclear assembly. It is usually denoted by $B_g^{\ 2}$

The term "buckling" is explained by the fact that $B^2 = -\dfrac{\nabla^2\phi}{\phi}$ is essentially the second derivative of ϕ divided by the function itself, which describes the curvatuve of ϕ

combining $\nabla^2\phi + B^2\phi = 0$ **and** $\nabla^2\phi + \dfrac{k_\infty e^{-B^2\tau} - 1}{L^2}\phi = 0$

$$k_{eff} = \frac{k_\infty e^{-B^2\tau}}{1 + L^2 B^2} = 1$$

Critical condition for a thermal reactor

7.4 Critical Equation and Reactor Buckling

$$k_{eff} = \frac{k_\infty e^{-B^2\tau}}{1+L^2B^2} = 1$$

determines B^2 in terms of the physical properties of the reactor materials which are involved through k_{eff} , τ , and L^2 .

The numerical value of B^2 as determined from this equation is called the **material buckling** of the reactor and designated by $B_m{}^2$.

Condition: the reactor is critical → $B_g{}^2 = B_m{}^2$

7.4 Critical Equation and Reactor Buckling

Important points:

In general, the choice of B^2 that satisfies the mathematical requirements of equation $\nabla^2\phi + B^2\phi = 0$ is not unique, but the smallest numerical is the one that has physical significance for our problem and it is this value which we shall be interested in obtaining.

B^2 has a dimension of cm^{-2}

Increasing the geometrical dimensions of a critical reactor causes the numerical value for the geometrical buckling $B_g{}^2$ to decrease.

Increasing the size of a reactor beyond its critical size results in a k_{eff} greater than unity.

Supercritical reactor for $k_{eff} > 1$

Subcritical reactor for $k_{eff} < 1$

7.5 The nonleakage Factors

$$Nonleakage\ \ factor = \frac{\text{actual neutron production rate per } cm^3}{\text{maximum rate for infinite reactor per } cm^3}$$

$$= \frac{k_\infty \Sigma_a \phi e^{-B^2\tau}}{k_\infty \Sigma_a \phi} = e^{-B^2\tau}$$

This is the fraction of fast neutrons that does not leak out of the assembly during slowing-down and reaches thermal energies

$$l_f = e^{-B^2\tau}$$

We can prove that the thermal nonleakage factor is: $l_{th} = \frac{1}{1+B^2L^2}$

Hence, the criticality equation become: $k_{eff} = k_\infty l_f l_{th} = \frac{k_\infty e^{-B^2\tau}}{1+L^2B^2}$

Example 9.1

7.6 Criticality of Large Thermal Reactors

It can be proven that for large reactors, B^2 becomes small

an expansion of the exponential term

$$k_{eff} = k_\infty \frac{e^{-B^2\tau}}{1+L^2B^2}$$
$$= k_\infty \frac{1}{e^{B^2\tau}(1+L^2B^2)} = k_\infty \frac{1}{(1+B^2\tau)(1+L^2B^2)} = \frac{k_\infty}{1+B^2(\tau+L^2)}$$

The quantity $L^2+\tau$ is usually denoted by M^2 and is known as the **migration area** and M as the **Migration length.**

$$M^2 = L^2 + \tau$$

For **large thermal critical reactor**

$$k_{eff} = \frac{k_\infty}{1+B^2M^2} = 1$$

Example 9.2, page 276

7.7 The critical Equation for Reactors with Hydrogeneous Moderators

It has already been mentioned that the Fermi age equation is not satisfactory for hydrogen or deuterium moderators. Why?

Answer: because the assumption of a large number of collisions preceding the attainment of thermal energies is not valid

On the contrary, the probability of thermalization by single neutron-moderator collisions is high.

If Σ^0 is the cross section for this type of collision, the factor for nonleakage probability during slowing down to be used, which has been found to be more satisfactory, is

$$l_{f(hydrogen)} = \frac{\tan^{-1}(B/\Sigma^0)}{B/\Sigma^0}$$

This expression replaces the factor $e^{-B^2\tau}$ in the criticality equation, which then becomes

$$k_{eff} = k_\infty \frac{\dfrac{\tan^{-1}(B/\Sigma^0)}{B/\Sigma^0}}{1+L^2B^2}$$

7.7 The critical Equation for Reactors with Hydrogeneous Moderators

The cross section Σ^0 can be determined experimentally from a measurement of τ_0

the expression : $\Sigma^0 = \dfrac{1}{`(3\tau_0)^{1/2}}$

7.8 Critical Size and Geometrical Buckling

The geometrical buckling was defined previously as the smallest number B^2 which satisfies equation

$$\nabla^2\phi + B^2\phi = 0$$

for the neutron flux in a critical reactor

Note that:

This differential equation is known as the wave equation

Solutions of the wave equation are obtained by first expressing the Laplacian in the variables that are best suited to the particular physical situation and then selecting the appropriate boundary conditions in order to get a solution

An obvious boundary condition for a **bare nuclear reactor** (reactor without reflector) would be the requirement that the neutron flux should be zero at the boundary of the reactor.

7.8 Critical Size and Geometrical Buckling

The shapes of nuclear reactors are almost exclusively either a rectangular parallelepiped, a sphere, or a cylinder.

The suitable coordinate systems are Cartesian for the first case, spherical for the second and cylindrical for the last one

Let us review the solution in different geometries

1) Parallelepiped of sides a,b,c

$$\nabla^2\phi + B^2\phi \rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} + B^2\phi = 0$$

The solution is :

$$\phi(x,y,z) = A\cos\left(\pi\frac{x}{a}\right)\cos\left(\pi\frac{y}{b}\right)\cos\left(\pi\frac{z}{c}\right) \qquad \textbf{(7.32)}$$

7.8 Critical Size and Geometrical Buckling

2) Sphere of Radius R

$$\nabla^2\phi + B^2\phi \rightarrow \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} + B^2\phi = 0 \qquad \textbf{(7.34)}$$

The solution is : $\phi(r) = \frac{A}{r}\sin\left(\pi\frac{r}{R}\right)$

3) Cylinder of height *H* and Base Radius *R*

$$\nabla^2\phi + B^2\phi \rightarrow \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} + B^2\phi = 0$$

The solution is : $\phi = A\cos\left(\pi\frac{z}{H}\right)J_0\left(\frac{\alpha r}{R}\right)$ **(7.36)**

Where J_0 is the zero order Bessel function and α is its smallest root $\alpha = 2.405$

7.8 Critical Size and Geometrical Buckling

By substituting the relevant solution back into the differential equation

$$\nabla^2\phi + B^2\phi = 0$$

we can find the geometrical buckling B^2 for a reactor of given shape and dimension

1) Spherical reactor

Can you prove this ?

We can prove that: $B^2 = \frac{\pi^2}{R^2}$ This relates the critical radius R to the geometrical buckling

→ **the critical radius is**: $R = \frac{\pi}{B}$ → critical volume V is :

$$V = \frac{4\pi}{3}R^3 = \frac{4\pi^4}{3B^3} = \frac{130}{B^3}$$

7.8 Critical Size and Geometrical Buckling

2) Cylinderical reactor

$$B^2 = \frac{\pi^2}{H^2} + \frac{(2.405)^2}{R^2}$$

To find the smallest critical volume of the cylinder we must minimize its volume $V = \pi R^2 H$

Eliminating R^2 → $$V = \frac{H^3 \cdot (2.405)^2}{B^2 H^2 - \pi^2}$$

Critical volume → $$\frac{dV}{dH} = 0$$ → $$B^2 = \frac{3\pi^2}{H^2}$$

Hence the minimum critical volume is : $$V_{\min} = \frac{148}{B^3}$$

7.8 Critical Size and Geometrical Buckling

3) parallelepiped reactor

By analogous procedure we can show that the minimum critical volume is a cube

a=b=c

$$B^2 = 3\frac{\pi^2}{a^2}$$

Note that for a given buckling,
the least critical volume is that of a sphere,
then a cylinder and then a cubic.

minimum critical volume is $V_{\min} = a^3 = 3^{3/2}\left(\frac{\pi^3}{B^3}\right)$

Note that: the geometrical buckling for a parallelepiped of sides *a, b and c* is given by

$$B^2 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{\pi^2}{c^2}$$

7.9 Extrapolation Length Correction

The neutron flux distributions as given by expressions 7.32, 7.34, and 7.36 for the various reactor shapes considered lead to a vanishing of the neutron flux at the reactor boundaries in accordance with the boundary conditions that had been postulated to apply in each case.

$$\phi(x,y,z) = A\cos\left(\pi\frac{x}{a}\right)\cos\left(\pi\frac{y}{b}\right)\cos\left(\pi\frac{z}{c}\right)$$

$$\phi(r) = \frac{A}{r}\sin\left(\pi\frac{r}{R}\right)$$

$$\phi = A\cos\left(\pi\frac{z}{H}\right)J_0\left(\frac{\alpha r}{R}\right)$$

In actual fact, the assumption of **zero flux at the boundary** is only approximately **true** for a **nuclear reactor**.

Indeed, the neutron flux at a boundary between a moderator where neutron diffusion is taking place and a vacuum (or air) can not be strictly zero, because there is always some neutron leakage flow from the moderator across the boundary into the vacuum or air.

7.9 Extrapolation Length Correction

When solving the differential equation 9.16 for a given reactor shape, it would therefore be more correct to use as the boundary condition for the neutron flux the condition that $\phi = \phi_0$, where ϕ_0 is the small but non-vanishing neutron flux at the boundary

See Figure 9.5, page 287

Instead of working with ϕ_0 it is more convenient to use a boundary condition for which the neutron flux does vanish nevertheless, and to reconcile the experimental fact of nonzero flux at the boundary with the mathematically desirable and convenient assumption of zero flux in the following manner:

We use a hypothetical boundary at a distance d beyond the actual physical boundary, where d is determined by a linear extrapolation of neutron flux from a point near the reactor boundary

This distance d beyond the physical boundary where the mathematical neutron flux vanishes is called the **extrapolation** or **augmentation distance.**

Transport theory lead to numerical value for this distance by the expression

$$d = 0.71\lambda_{tr} = 0.71\frac{\lambda_s}{1-\left(2/3A\right)}$$

7.9 Extrapolation Length Correction

This extrapolation does not correspond to a physical condition of the system, but is merely a mathematical device to obtain a convenient boundary condition for the solution of the diffusion equation.

As a consequence of the introduction of the extrapolated distance, the critical dimensions which were obtained in our critical size calculations are not the physical dimensions of the material but include the extrapolation distance

To obtain the physical dimensions, the extrapolation distance must be subtracted from each moderator vacuum boundary. See Figures 9.6 and 9.7.

If R and a are the critical distances as obtained from the diffusion equation, and R_m and a_m are the corrected material distances, then according to Figure 9.7:

$$2R_m = 2R - 2d$$

$$\begin{aligned} R_m &= R - d = R - 0.71\lambda_{tr} \\ &= R - 0.71\frac{\lambda_s}{1-\left(2/3A\right)} \end{aligned}$$

And according to Fig.9.6

$$\begin{aligned} a_m &= a - 2d \\ &= a - 1.42\frac{\lambda_s}{1-\left(2/3A\right)} \end{aligned}$$

7.10 Effect of Reflector

Question: what is a reflector?

The effect of neutron leakage can be reduced by the employment of a neutron **reflector** which surrounds the active core of the reactor and which reflects or scatters the escaped neutrons back into the core.

The qualifications of an efficient reflector are very much the same as those for a good moderator. The efficiency of a reflector are very much the same as those for a good moderator.

The efficiency of a reflector is described by its **albedo-** a term borrowed from astronomy- which is the fraction of neutrons reflected back into the reactor out of all the neutrons incident on the reflector, i.e., the ratio of reflected to incident neutrons**.**

finite reactor without a reflector- generally called a **bare reactor**

7.10 Effect of Reflector

Surrounding a reactor with a reflector which is a medium of high scattering cross section and low absorption cross section, has certain distinct advantages over the bare reactor which are briefly stated as follows:

1) Improved neutron economy:
the reflector reduces neutron leakage from the core by reflecting or scattering many of the escaping neutrons back into the core region of the reactor and also acts as a moderator for fast neutrons that have entered it from the core.
in a way, the moderation of the fast neutrons in the reflector will be more efficient than in the core itself, since the absence of neutron absorbing material in the reflector will reduce neutron loss due to resonance escape absorption, so that a large fraction of the fast neutrons in the reflector can reach thermal energies than is possible in the fuel-containing region of a moderator. Look at Figure 9.8

2) Possibility of fuel savings:
the improvement in the neutron economy reduces the amount of fuel or the fuel concentration required to achieve criticality. In other words, fuel saving can be achieved if a reflector is incorporated in the design of a nuclear reactor.

7.10 Effect of Reflector

<u>3) Improved reactor power utilization</u>:
the improvement in the power utilization is a consequence of the flux flattening across the reactor core that occurs when a reflector is incorporated (See Fig. 9.8).
the neutron flux will be markedly greater at the core-reflector interface than its value there, i.e., at the boundary of the bare reactor. Thus the average neutron flux is greater for a reflector reactor than for a bare reactor with the same maximum neutron flux.

Since the power production rate is proportional to the average netron flux, the reactor can be operated at a higher total power output for the same maximum neutron flux. By virtue of the flux flattening effect the power production rate is more uniform over the core volume, which is highly desirable from the operational point of view, especially with large power reactors.

Homework

- Problems:2, 3, 5,6, 7, 9 of Chapter 9 in Text Book, Pages 292

- الى اللقاء في الحصة القادمة ان شاء الله